

Perturbative QCD in acceptable schemes with holomorphic coupling*

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Perturbative QCD in mass independent schemes leads in general to running coupling $a(Q^2)$ which is nonanalytic (nonholomorphic) in the regime of low spacelike momenta $|Q^2| \lesssim 1 \text{ GeV}^2$. Such (Landau) singularities are inconvenient in the following sense: evaluations of spacelike physical quantities $\mathcal{D}(Q^2)$ with such a running coupling $a(\kappa Q^2)$ ($\kappa \sim 1$) give us expressions with the same kind of singularities, while the general principles of local quantum field theory require that the mentioned physical quantities have no such singularities. In a previous work, certain classes of perturbative mass independent beta functions were found such that the resulting coupling was holomorphic. However, the resulting perturbation series showed explosive increase of coefficients already at N⁴LO order, as a consequence of the requirement that the theory reproduce the correct value of the τ lepton semihadronic strangeless decay ratio r_τ . In this work we successfully extend the construction to specific classes of perturbative beta functions such that the perturbation series do not show explosive increase of coefficients, the perturbative coupling is holomorphic, and the correct value of r_τ is reproduced. In addition, we extract, with Borel sum rule analysis of the $V + A$ channel of the semihadronic strangeless decays of τ lepton, reasonable values of the corresponding $D = 4$ and $D = 6$ condensates.

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I. INTRODUCTION

The perturbative QCD (pQCD) calculations are usually performed in mass independent schemes, i.e., schemes in which beta function $\beta(a)$ of the running coupling (couplant) $a(Q^2)$ ($\equiv \alpha_s(Q^2)/\pi$) has expansion in powers of a such that the beta expansion coefficients depend on the number of effective quark flavors N_f . When the squared momenta $q^2 \equiv -Q^2$ are low, $|Q^2| \lesssim 1 \text{ GeV}^2$, the mentioned coefficients have $N_f = 3$. Such calculations give for the running coupling $a(Q^2)$ a function which has, for the general spacelike momenta $Q^2 \in \mathbb{C} \setminus (-\infty, 0]$, nonholomorphic (singular) behavior in the small momentum regime $|Q^2| \lesssim 1 \text{ GeV}^2$, and these singularities are usually called Landau ghosts (or Landau singularities). When any spacelike physical quantities $\mathcal{D}(Q^2)$, such as the current correlators and structure functions, are evaluated in pQCD as (truncated) series involving such coupling $a(\kappa Q^2)$ (where $\kappa \sim 1$ is a positive renormalization scale parameter), the resulting expressions $\mathcal{D}(Q^2)_{\text{eval.}} = \mathcal{F}(a(\kappa Q^2))$ manifest the same type of singularities for $Q^2 \in \mathbb{C} \setminus (-\infty, 0]$ (and $|Q^2| \lesssim 1 \text{ GeV}^2$). Such singularities of $\mathcal{D}(Q^2)_{\text{eval.}}$ are physically unacceptable, because $\mathcal{D}(Q^2)$ must be an analytic (holomorphic) function of Q^2 in the entire complex Q^2 plane with the exception of the negative semiaxis $Q^2 \in \mathbb{C} \setminus (-\infty, -M_{\text{thr}}^2]$ (where the threshold mass is $M_{\text{thr}} \sim 0.1 \text{ GeV}$), this being a consequence of general principles of (local) quantum field theories [1, 2]. Even resummations of infinite number of terms in the perturbation expansion of $\mathcal{D}(Q^2)$, practicable in QCD for example in the large- β_0 approximation, do not cure the problem of Landau ghosts [cf. comments following Eqs. (A11) in Appendix A]. If we are to apply a universal running coupling $a(Q^2)$ in the evaluation of a low-momentum quantity $\mathcal{D}(Q^2)$, the analytic properties of $a(Q^2)$ should reflect the mentioned analytic properties of $\mathcal{D}(Q^2)$.

The notion of a universal running coupling $a(Q^2)$ is intimately connected with the concept of perturbation expansion. Since perturbation theory is directly applicable only to those physical quantities or, to those circumstances (momenta, etc.), which are characterized by small coupling, originally only such coupling makes direct sense. Within QCD this is the coupling in the regime of high momenta (asymptotic freedom) where partons (quarks and gluons) do exist in the usual sense. Nevertheless, one can attribute a meaning to a universal running coupling outside the high momentum regime. One of the preconditions for the applicability of such a coupling is that the aforementioned nonanalyticity (Landau singularities) of $a(Q^2)$, at low $|Q^2| \lesssim 1 \text{ GeV}^2$ in the complex Q^2 plane outside the negative semiaxis, does not appear or is eliminated.

A formalism exists which extends the use of the universal running coupling to the regime $|Q| \sim 1 \text{ GeV}$, namely the Operator Product Expansion (OPE) in the sense of the ITEP School (pQCD+OPE), Refs. [3, 4]. In such approach

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the inclusive spacelike quantities $\mathcal{D}(Q^2)$, are evaluated by adding to the usual perturbation expansion of the term with the lowest dimension (leading-twist term), $\mathcal{F}(a(\kappa Q^2))$, other terms which involve vacuum expectation values (condensates) of various operators O_N with higher dimensions $2N$, i.e., terms proportional to $\langle O_N \rangle / Q^{2N}$. Complete formalism which would extend the regime of applicability of this pQCD+OPE approach at present does not exist, but attempts have been made in this direction with the use of nonlocal condensates [5].

Various independent lines of research support the existence of the concept of the running coupling $a(Q^2)$ in low-momentum regime and suggest that it is finite and possibly holomorphic there: the Gribov-Zwanziger approach [6–9]; calculations involving Dyson-Schwinger equations (DSE) for gluon and ghost propagators and vertices [10–21]; stochastic quantization [22]; functional renormalization group equations [23–25]; lattice calculations [26–29]. In addition, the finiteness of the coupling at $Q \rightarrow 0$ is suggested by specific applications [30, 31] of the Principle of Minimal Sensitivity (PMS) [32–34], by models using the AdS/CFT correspondence modified by a dilaton background [35, 36], in scenarios with larger quark flavor number N_f [37, 38], and in various other approaches such as those in Refs. [39–47].

The nonanalyticity of $a(Q^2)$ in low-momentum regimes in the usual pQCD schemes was addressed in the seminal works of Shirkov, Solovtsov *et al.* [48–52], where a holomorphic version $\mathcal{A}^{(\text{APT})}(Q^2)$ of the pQCD coupling $a(Q^2)$ (in any mass independent scheme) was constructed, via a use of the Cauchy theorem and dispersion integral in which the offending (Landau) cut of $a(Q^2)$ was eliminated and the cut of $a(Q^2)$ for $Q^2 < 0$ was left unchanged; in a sense, this is a “minimal” analytization approach, widely referred in the literature as Analytic Perturbation Theory (APT). This approach includes the analogous construction, via dispersive integral, of the holomorphic analogs $\mathcal{A}_n^{(\text{APT})}(Q^2)$ of the (integer) powers $a(Q^2)^n$ of pQCD coupling. The formalism was later extended to the construction of APT analogs of any physical quantity [53], and of APT analogs $\mathcal{A}_\nu^{(\text{FAPT})}(Q^2)$ of noninteger powers $a(Q^2)^\nu$ (ν noninteger) in the works [54–56] (Fractional Analytic Perturbation Theory - FAPT). For a review of FAPT, see Refs. [57, 58], and mathematical packages for numerical calculation are given in Refs. [59–61].

Since the publication of APT [48–50], several other (extended) analytic QCD models, i.e., models of holomorphic $\mathcal{A}(Q^2)$, have been constructed [62–75].¹ Analytic QCD models [(F)APT and others] and related dispersive approaches have been used in various contexts [76–90]. For reviews of some analytic QCD models, see Refs. [91, 92].

Furthermore, the higher power analogs $\mathcal{A}_\nu(Q^2)$ of $a(Q^2)^\nu$ in such general analytic QCD models are constructed by the procedures of Refs. [70, 71] (when ν is integer) and Ref. [93] (when ν is general real).

It turns out that all these holomorphic couplings $\mathcal{A}(Q^2)$ are nonperturbative, i.e., for $|Q^2| > \Lambda^2$ (where $\Lambda^2 \sim 1 \text{ GeV}^2$) they differ from the corresponding pQCD coupling $a(Q^2)$ (i.e., $a(Q^2)$ in the same scheme) by terms $\mathcal{A}(Q^2) - a(Q^2) \sim (\Lambda^2/Q^2)^N$, where $N = 1$ in the models of Refs. [48, 49, 62–68, 70, 71, 75], $N = 3, 4, 5$ in the models of Refs. [72, 73], [69], and [74], respectively. However, the power terms Λ^2/Q^2 , at high $|Q^2|$ (small $a(Q^2)$), can be expressed as $\exp[-1/(\beta_0 a(Q^2))] \sim e^{-1/a}$, which is a nonanalytic function in a (around $a = 0$). This implies that the analytic QCD models cannot be described by a perturbative beta function $\beta(a) \equiv da(Q^2)/d\ln Q^2$, i.e., by a $\beta(a)$ function which is described at small $|a|$ fully by its Taylor expansion in powers of a . The function $\beta(\mathcal{A})$ in all these analytic QCD models contains terms $\sim e^{-1/\mathcal{A}}$.

In this context, the following question appears naturally: does there exist a perturbative $\beta(a)$ function [$\beta(a) = -\beta_0 a^2 - \beta_1 a^3 - \beta_2 a^4 - \dots$] such that the corresponding (perturbative) running coupling $a(Q^2)$ is a holomorphic function in the complex plane $Q^2 \in \mathbb{C} \setminus (-\infty, -M_{\text{thr}}^2]$ (or $Q^2 \in \mathbb{C} \setminus (-\infty, 0]$)? In Refs. [94, 95], an extensive attempt was made to obtain such an analytic pQCD (anpQCD). The major obstacles to such an effort turned out to be the simultaneous fulfillment of two requirements: a) $a(Q^2)$ is holomorphic; b) the value of the best measured low-energy QCD observable $r_\tau = 0.203 \pm 0.004$ can be reproduced in this anpQCD. Here, r_τ is the QCD massless canonical part [$r_\tau = a + \mathcal{O}(a^2)$] of the $(V + A)$ -channel of the τ lepton strangeless semihadronic decay ratio $R_\tau(\Delta S = 0)$, and in r_τ the quark mass effects have been subtracted and the chirality-conserving higher-twist effects are known to be very suppressed [96]. The two requirements a) and b) have the tendency to be mutually exclusive: almost any anpQCD gives far too low value (< 0.14) of r_τ ; if the free parameters in the considered classes of perturbative $\beta(a)$ functions are varied in such a way that the value 0.203 of r_τ is approached (from below), the coupling $a(Q^2)$ in general acquires singularities inside the plane $Q^2 \in \mathbb{C} \setminus (-\infty, 0]$ and thus ceases to be holomorphic. The problem of too low value of r_τ was already encountered earlier [77, 78, 97] for the analytic (and nonperturbative) QCD model APT of Refs. [48–52]. Nonetheless, in Refs. [94, 95], for specific classes of perturbative $\beta(a)$ functions with holomorphic $a(Q^2)$, the $\beta(a)$ functions were modified/multiplied by another perturbative function $f_{\text{fact}}(a)$ such that the perturbation expansion of r_τ , including its first four (known) terms, gave the correct value 0.203 and the analyticity of $a(Q^2)$ was preserved. However, the price to pay was that the resulting beta function acquired in its expansion very large β_4 coefficient at a^5 ($\beta_4 \sim 10^6$ – 10^7) and thus the fifth term $\sim a^5$ in the expansion of r_τ became uncontrollably high.

¹ In Refs. [62–65] the coupling is holomorphic for $Q^2 \in \mathbb{C} \setminus (-\infty, 0]$ and is infinite at $Q^2 = 0$.

In this work we return to this problem and find an attractive solution to the mentioned problem, by constructing such perturbative $\mathcal{F}_{\text{fact}}(a)$ functions that give perturbative beta functions $\beta(a) \propto \mathcal{F}_{\text{fact}}(a)$ that simultaneously: (a) keep the perturbation expansion coefficients under control to an arbitrarily high order; (b) reproduce the correct value $r_\tau = 0.203$; (c) preserve the analyticity of $a(Q^2)$ in $Q^2 \in \mathbb{C} \setminus (-\infty, -M_{\text{thr}}^2]$. In Sec. II we present the formalism of integration of the renormalization group equation in the complex Q^2 plane and various conditions (analyticity, universality) that have to be fulfilled. In Sec. III we reproduce several classes of β functions that give holomorphic $a(Q^2)$ but fail to achieve the value of $r_\tau = 0.203$. In Sec. IV we introduce the functions $\mathcal{F}_{\text{fact}}(a)$ with which we modify/multiply the beta functions of the previous Section and which give us the acceptable perturbative analytic QCD framework: holomorphic $a(Q^2)$, the correct value $r_\tau = 0.203$, and the perturbation expansion coefficients under control. In Sec. V we perform, with one of the obtained perturbative analytic QCD schemes, an analysis with Borel sum rules of the $V + A$ channel of the semihadronic decays of τ lepton and extract reasonable values of the corresponding $D = 4$ and $D = 6$ condensates. In Sec. VI we summarize our results.

II. CONDITIONS, INTEGRATION

The running coupling $a(Q^2) \equiv \alpha_s(Q^2)/\pi$ in QCD fulfills the renormalization group equation (RGE)

$$Q^2 \frac{da(Q^2)}{dQ^2} = \beta(a(Q^2)) , \quad (1)$$

where $\beta(a)$ is beta function. In the approach of the construction of the perturbative and holomorphic coupling $a(Q^2) = \mathcal{A}(Q^2)$ here, the starting point will be the construction of beta function $\beta(a)$, and then the coupling function $a(Q^2)$ will be obtained by numerical integration of the RGE (1) in the complex Q^2 plane. We will impose three central requirements on $\beta(a)$ and the resulting $a(Q^2)$ functions:

1. The coupling $a(Q^2)$ is a perturbative (pQCD coupling); this is equivalent to the requirement that beta function is a holomorphic (analytic) function of a at $a = 0$

$$\beta(a) = -\beta_0 a^2 - \beta_1 a^3 - \beta_2 a^4 - \dots = -\beta_0 a^2 (1 + c_1 a + c_2 a^2 + \dots) , \quad (2)$$

cf. Refs. [94, 95, 98–100]. For example, $\beta(a)$ cannot contain the typically nonperturbative terms $\sim \exp[-C/a(Q^2)]$ for which the Taylor expansion around $a = 0$ is blind.

2. The coupling $a(Q^2)$ must reproduce the correct measured value $r_\tau = 0.203 \pm 0.004$, where r_τ is the QCD massless canonical part [$r_\tau = a + \mathcal{O}(a^2)$] of τ lepton strangeless semihadronic decay ratio $R_\tau(\Delta S = 0)$ (with the quark mass effects subtracted and the higher-twist effects suppressed). We recall that r_τ is at the moment the best measured inclusive low-energy QCD observable.
3. The coupling $a(Q^2)$, constructed by the integration of the RGE (1), must be a holomorphic function, i.e., holomorphic in the complex Q^2 plane $Q^2 \in \mathbb{C} \setminus (-\infty, -M_{\text{thr}}^2]$, where the threshold mass is $M_{\text{thr}} \sim 0.1$ GeV.

In the integration of Eq. (1), we first need the initial condition $a(Q_{\text{in}}^2)$ at an initial (low) scale Q_{in}^2 . Since we are interested in the holomorphic behavior of $a(Q^2)$ at not very high $|Q^2|$ ($|Q^2| \lesssim (3m_c)^2$), we consider for simplicity the heavy quarks c, b, t to be decoupled, and the three light quarks u, d, s we will consider to be massless. Stated otherwise, the number of active flavors in the RGE (1) is $N_f = 3$. We choose our initial scale to be $Q_{\text{in}}^2 = (3m_c)^2 \approx 14.52 \text{ GeV}^2$. In order to obtain the value of $a(Q_{\text{in}}^2)$, i.e., in the scheme determined by the considered beta function $\beta(a)$, we should first obtain the value $\bar{a}(Q_{\text{in}}^2)$ in the $\overline{\text{MS}}$ scheme. For this, we take the present world average [101] $\bar{a}(M_Z^2) = 0.1184/\pi$ and RGE-run it down by the known 4-loop $\overline{\text{MS}}$ beta function $\bar{\beta}(a)$ from $Q^2 = M_Z^2$ to $Q_{\text{in}}^2 = (3m_c)^2$. The quark thresholds are taken at $Q_{\text{thr}} = \kappa \bar{m}_b$ and $Q_{\text{thr}} = \kappa \bar{m}_c$, according to the 3-loop matching conditions [102], where $1 \leq \kappa \sim 1$ and we take $\kappa = 3$, and the $\overline{\text{MS}}$ masses $\bar{m}_b = 4.20 \text{ GeV}$ and $\bar{m}_c = 1.27 \text{ GeV}$. This gives us $\bar{a}(Q_{\text{in}}^2) \approx 0.0716$ (at $N_f = 3$). The corresponding value of $a(Q_{\text{in}}^2)$ is then obtained by using the integrated form of RGE (i.e., implicit solution) in its subtracted form, cf. Appendix A of Ref. [32] (cf. also Appendix A of Ref. [103])

$$\begin{aligned} \frac{1}{a} + c_1 \ln \left(\frac{c_1 a}{1 + c_1 a} \right) + \int_0^a dx \left[\frac{\beta(x) + \beta_0 x^2 (1 + c_1 x)}{x^2 (1 + c_1 x) \beta(x)} \right] \\ = \frac{1}{\bar{a}} + c_1 \ln \left(\frac{c_1 \bar{a}}{1 + c_1 \bar{a}} \right) + \int_0^{\bar{a}} dx \left[\frac{\bar{\beta}(x) + \beta_0 x^2 (1 + c_1 x)}{x^2 (1 + c_1 x) \bar{\beta}(x)} \right] , \end{aligned} \quad (3)$$

where $a \equiv a(Q_{\text{in}}^2) = a_{\text{in}}$ and $\bar{a} \equiv \bar{a}(Q_{\text{in}}^2) \approx 0.0716$. We note that $\beta_0 = (1/4)(11 - 2N_f/3)$ and $c_1 = \beta_1/\beta_0 = (1/4)(102 - 38N_f/3)/(11 - 2N_f/3)$ are the universal beta-function coefficients in the mass independent schemes

($\beta_0 = 9/4$ and $c_1 = 16/9$ for $N_f = 3$), while the other expansion coefficients $c_j \equiv \beta_j/\beta_0$ ($j \geq 2$) in Eq. (2) characterize the scheme [32]. Any choice of β function then determines, via Eq. (3), the initial value $a(Q_{\text{in}}^2) = a_{\text{in}}$ for the numerical integration of Eq. (1).²

As mentioned, the RGE (1) will be solved numerically not just for $Q^2 > 0$, but in the entire complex Q^2 plane. Following the presentation in Refs. [94, 95], a new complex variable is introduced: $z = \ln(Q^2/Q_{\text{in}}^2)$. Then the first sheet of the complex Q^2 plane corresponds to the semiopen stripe $-\pi \leq \text{Im}z < +\pi$ in the complex z plane. The general spacelike regime $Q^2 \in \mathbb{C} \setminus (-\infty, 0]$, where $a(Q^2)$ is holomorphic in the considered perturbative analytic QCD (anpQCD) scenarios, is represented by the open stripe $-\pi < \text{Im}(z) < +\pi$ in the z -plane. The timelike (Minkowskian) semiaxis $Q^2 \leq 0$ corresponds to the border line $\text{Im}z = -\pi$ of the z -stripe. The point $Q^2 = 0$ is $z = -\infty$, and $Q^2 = Q_{\text{in}}^2$ ($\approx 14.52 \text{ GeV}^2$) is $z = 0$. In Figs. 1 (a) and (b) we present the corresponding general spacelike and timelike regimes in the complex Q^2 plane and on the z stripe, respectively, with a view that $a(Q^2)$ is holomorphic in the extended regime $Q^2 \in \mathbb{C} \setminus (-\infty, -M_{\text{thr}}^2]$ (where $0 \leq M_{\text{thr}} \lesssim 0.1 \text{ GeV}$). Let us denote $a(Q^2) \equiv F(z)$. Then RGE (1) acquires the

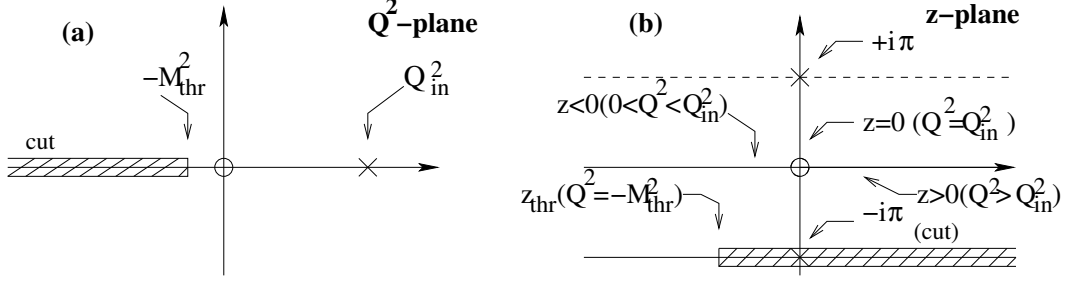


FIG. 1: (a) Complex Q^2 plane; (b) complex z plane where $z = \ln(Q^2/Q_{\text{in}}^2)$; the non-timelike stripe is $-\pi \leq \text{Im}z < +\pi$.

form

$$\frac{dF(z)}{dz} = \beta(F(z)) \quad (F(z) \equiv a(Q^2)) \quad (4)$$

in terms of $z = \ln(Q^2/Q_{\text{in}}^2)$ in the semiopen stripe $-\pi \leq \text{Im}z < +\pi$. The requirement that $a(Q^2)$ be holomorphic for $Q^2 \in \mathbb{C} \setminus (-\infty, 0]$ now means that $F(z)$ is holomorphic ($\Rightarrow \partial F/\partial \bar{z} = 0$) in the open stripe $-\pi < \text{Im}(z) < +\pi$. The (physical) singularities can appear only on the timelike line $\text{Im}(z) = -\pi$. Let us denote $z = x + iy$ and $F = u + iv$; then we can rewrite RGE (4) as a coupled system of real partial differential equations for $u(x, y)$ and $v(x, y)$

$$\frac{\partial u(x, y)}{\partial x} = \text{Re}\beta(u + iv), \quad \frac{\partial v(x, y)}{\partial x} = \text{Im}\beta(u + iv), \quad (5a)$$

$$\frac{\partial u(x, y)}{\partial y} = -\text{Im}\beta(u + iv), \quad \frac{\partial v(x, y)}{\partial y} = \text{Re}\beta(u + iv). \quad (5b)$$

We recall that $x = \ln(|Q^2|/Q_{\text{in}}^2)$, $y = \arg(Q^2)$ ($-\pi \leq y < \pi$), $u = \text{Re } a$, $v = \text{Im } a$. Having chosen an Ansatz for beta function $\beta(a(Q^2)) \equiv \beta(F(z))$ and the corresponding initial condition value $a(Q_{\text{in}}^2)$, the integration of equations (5) is then implemented numerically to high precision by the MATHEMATICA software [104]. Numerical analyses suggest that it is very difficult to obtain in this way analytic coupling $a(Q^2)$ for $Q^2 \in \mathbb{C} \setminus (-\infty, 0]$, i.e., analytic $F(z)$ in the entire open stripe $-\pi < \text{Im}z < \pi$, unless we require in addition also analyticity in and around the point $Q^2 = 0$ ($z = -\infty$). Stated otherwise, with certain classes of pQCD β -functions we obtain the correct holomorphic behavior of $a(Q^2)$. We represent the analyticity in Q^2 at $Q^2 = 0$ in the form

$$a(Q^2) = a_0 + a_1(Q^2/\Lambda^2) + \mathcal{O}[(Q^2/\Lambda^2)^2], \quad (6)$$

where $a_0 = a(Q^2 = 0) = F(z = -\infty) < \infty$, and $a_1 \neq 0$. Applying the derivative $d/dz = d/d \ln Q^2$ to this series, the condition reads

$$\beta'(F)|_{F=a_0} = +1. \quad (7)$$

² Eq. (3) states that the left-hand side is exactly independent of the renormalization scheme parameters c_j ($j \geq 2$) appearing in the expansion (2) of $\beta(x)$; this (exact) independence was proven in Appendix A of Ref. [32].

The Ansätze for beta function are thus taken in the form (cf. Refs. [94, 95])

$$\beta(F) = -\beta_0 F^2 (1 - Y) f(Y)|_{Y \equiv F/a_0}, \quad (8)$$

where the function $f(Y)$ fulfills three conditions

$$f(Y) \quad \text{analytic at } Y = 0 \quad (\text{pQCD}), \quad (9a)$$

$$f(Y) = 1 + (1 + c_1 a_0) Y + \mathcal{O}(Y^2) \quad (\text{pQCD}), \quad (9b)$$

$$a_0 \beta_0 f(1) = 1 \quad (Q^2 = 0 \text{ analyticity}). \quad (9c)$$

The first condition says that the beta function is perturbative; the second accounts for the universality of the c_1 coefficient of the pQCD expansion (2); the third condition is the aforementioned condition of analyticity of $a(Q^2)$ at $Q^2 = 0$, i.e., Eq. (6). Let us once more recall that the choice of a specific type of beta function corresponds to a specific renormalization scheme, characterized by the coefficients c_j ($j \geq 2$) of the series expansion (2) of this beta function.

If $a_1 = 0$ in Eq. (6) and $f(Y)$ is any rational (Padé) function, Landau singularities appear, as argued in Ref. [95] (footnote 3 and Appendix A there); e.g., when $a(Q^2) = a_0 + \mathcal{O}((Q^2/\Lambda^2)^n)$ with $n \geq 2$, Landau poles of $F(z)$ appear at $\text{Im } z = \pm\pi/n$.

The condition of analyticity at $Q^2 = 0$, i.e. Eq. (6), implies that there is a finite region of analyticity of $a(Q^2)$ around $Q^2 = 0$, i.e., that the branching point of the cut $(-\infty, -M_{\text{thr}}^2]$ of $a(Q^2)$ in the complex Q^2 plane starts at a nonzero threshold energy $-M_{\text{thr}}^2 < 0$. This implicitly signals that the masses of light pseudoscalar mesons π and K are nonzero, i.e., that the masses of u , d and s quarks are not strictly zero. Therefore, the condition (6) implicitly incorporates these effects, which would otherwise be very difficult to incorporate explicitly with nonzero light quark masses in the RGE. Another, more practical, reason for imposing the condition (6) lies in the fact that it turned out to be very difficult or impossible to achieve numerically analyticity of $a(Q^2)$ in the Euclidean complex plane $Q^2 \in \mathbb{C} \setminus (-\infty, 0]$ unless the point $Q^2 = 0$ was also included as a point of analyticity of $a(Q^2)$, Refs. [94, 95].

Often in pQCD, the PMS [32–34] and effective charge (ECH) [105–110] schemes (at n -loop level, n finite) are constructed from a truncated perturbation series $\mathcal{D}_{\text{pt}}(Q^2)^{[n]}$ (i.e., including the terms up to a^n) of a considered spacelike observable $\mathcal{D}(Q^2)$ in such a way that, in the PMS procedure all the terms $\sim a^{n+2}$ are consistently discarded in the derivatives $\partial \mathcal{D}(Q^2)_{\text{PMS}}^{[n]} / \partial \text{RS} = 0$ (where $\text{RS} = \ln \mu^2, c_2, c_3, \dots$), and in the ECH procedure all the terms $\sim a^{n+1}$ are consistently discarded in $\mathcal{D}(Q^2)_{\text{ECH}}^{[n]}$ (for example, cf. Refs. [111]). Such schemes have scheme coefficients c_j ($j = 2, \dots, n-1$) which are independent of Q^2 of the considered observable $\mathcal{D}(Q^2)$. If such (PMS or ECH) schemes give finite $a_0 \equiv a(0)$, e.g. those with $c_2 < 0$, they in general do not result in holomorphic coupling $a(Q^2)$, at least not at $Q^2 = 0$, because a_0 in general does not fulfill the condition (9c). If $a_0 \beta_0 f(1) > 1$, there are Landau singularities and poles inside the z stripe (cf. Appendix A of Ref. [95]); if $a_0 \beta_0 f(1) < 1$, it can happen that no singularities appear inside the z stripe and the only point of nonanalyticity is $Q^2 = 0$; but then the value of r_τ is even generally much more below the experimental value $r_\tau = 0.203$, Ref. [112].

III. BETA FUNCTIONS AND RESULTS

Among $f(Y)$ functions that satisfy the three conditions (9), only certain specific subsets, with free parameters within $f(Y)$ varying in restricted intervals, lead upon the numerical integration of RGE's (5) to holomorphic behavior, i.e., to a holomorphic $F(z)$ in the entire open stripe $-\pi < \text{Im } z < \pi$. However, the evaluation of the aforementioned τ lepton decay ratio r_τ gave us consistently values well below the experimental values 0.203 ± 0.004 , namely values below 0.15. In Refs. [94, 95], for representation of the numerical results, various Ansätze were used for the function $f(Y)$: (1) in the form of polynomials; (2) Padé's (ratios of polynomials); (3) product of rescaled and translated functions of the type $(e^{-Y} - 1)/Y$ and $Y/(e^{-Y} - 1)$, respectively. As mentioned, it turned out that, while such functions did give us holomorphic $F(z)$ in the entire open stripe of z , they gave for r_τ far too low values (< 0.15). Here we summarize some of the results of Refs. [94, 95] for the three mentioned cases.

1. The case of quadratic polynomial $f(Y)$

$$f(Y) = 1 + r_1 Y + r_2 Y^2, \quad (10)$$

where the first coefficient is $r_1 = (1 + c_1 a_0)$ due to the condition (9b). In order to see whether the resulting running coupling $a(Q^2) = F(z) \equiv F(x + iy)$ has or has not singularities within the physical stripe $-\pi < z < \pi$ (Landau singularities), we present in Figs. 2(a) and (b) the results for the quantity $|\beta(F(z))|$ which should manifest singularities at the same z values as the singularities of $F(z)$. In the case $r_2 = 0$, Fig. 2(a) suggests

that there are no Landau singularities, i.e., no singularities on the open stripe $-\pi < \text{Im}z < \pi$, only singularities on the timelike axis ($\text{Im}z = \pm\pi$). In the case $r_2 < 0$ ($r_2 = -2$ taken), Fig. 2(b) clearly shows that there are Landau singularities. As argued in Ref. [95], for $0 \leq r_2 < r_1^2/4$ there are no Landau poles. In the $r_2 = 0$ case there are no free parameters, because the apparently free parameters r_1 and $a_0 = a(Q^2 = 0)$ are fixed by the conditions (9b)-(9c): $a_0 = 0.1901$ and $r_1 = (1 + c_1 a_0) = 1.338$. We did not choose $r_2 > 0$ because, although the coupling is holomorphic, the resulting r_τ is even lower than in the $r_2 = 0$ case. We refer to the case (10) with $r_2 = 0$ as P[1/0] because $F(Y)$ is Padé P[1/0](Y) in this case.³

2. The case of Padé [1/1] $f(Y)$

$$f(Y) = \frac{(1 - t_1 Y)}{(1 - u_1 Y)}. \quad (11)$$

We have seemingly three parameters (t_1 , u_1 and a_0), but two of them are eliminated by the conditions (9b)-(9c). We can regard as the only free parameter the coefficient u_1 . It turns out that for $u_1 = -0.1$ we obtain approximately largest r_τ while still no Landau poles.

3. The case of $f(Y)$ being a product of rescaled and translated functions of the type $(e^{-Y} - 1)/Y$ and $Y/(e^{-Y} - 1)$

$$\text{EE} : f(Y) = \frac{(\exp[-k_1(Y - Y_1)] - 1)}{[k_1(Y - Y_1)]} \frac{[k_2(Y - Y_2)]}{(\exp[-k_2(Y - Y_2)] - 1)} \times \mathcal{K}(k_1, Y_1, k_2, Y_2). \quad (12)$$

Here, the constant \mathcal{K} ensures the required normalization $f(Y = 0) = 1$. At first sight, we have five free parameters: $a_0 \equiv a(Q^2 = 0)$ and four parameters for translation and rescaling (Y_1 , k_1 , Y_2 , k_2). Two of the parameters (Y_2 and a_0) are eliminated by the conditions (9b)-(9c). Further, $0 < k_1 < k_2$ must be fulfilled to get physically acceptable behavior. Figs. 3(a), (b) represent the numerical results for $|\beta(F(z))|$ for the following two chosen cases: (a) $Y_1 = 0.1$; $k_1 = 10$; $k_2 = 11$ ($\Rightarrow Y_2 \approx 0.1839$); (b) $Y_1 = 1.1$; $k_1 = 6$; $k_2 = 11$ ($\Rightarrow Y_2 \approx 0.2386$). Figs. 3 suggest that the case (a) has no Landau singularities, and that the case (b) clearly has Landau singularities. The case EE(a) is such that $a(Q^2)$ is kept holomorphic and simultaneously the value of r_τ is higher than for most of other choices of EE parameters (but still not high enough, see later).

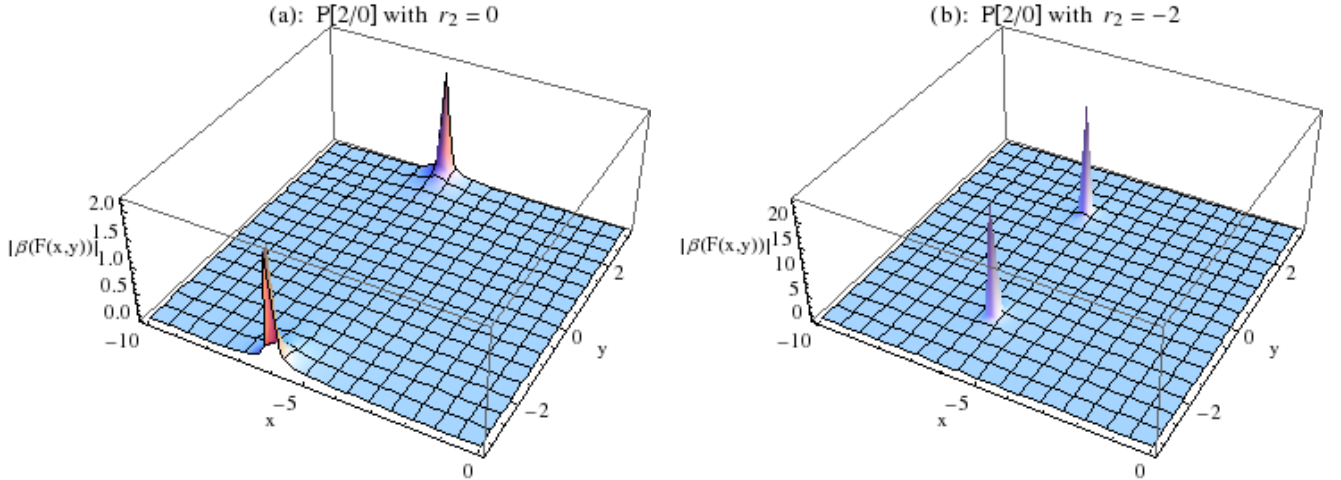


FIG. 2: $|\beta(F(z))|$ as a function of $z = x + iy$ for the beta function (8) with $f(Y)$ having the form (10) with: (a) $r_2 = 0$; (b) $r_2 = -2$.

³ The model of Ref. [30, 31], based on the principle of minimal sensitivity (PMS) [32–34], has the same form of beta function, with the conditions (9a)-(9b) fulfilled, but $a_0 \equiv a(0)$ does not satisfy the condition (9c), which in this case states: $a_0 \beta_0 (2 + c_1 a_0) = 1$. Namely, the model of Ref. [30, 31] has $a_0 \beta_0 (2 + c_1 a_0) > 1$, the coupling is thus not analytic at $Q^2 = 0$. In the version of the PMS approach applied in Ref. [30, 31], the resulting scheme coefficients depend on the squared momentum Q_0^2 of the considered observable $\mathcal{D}(Q_0^2)$. It is possible that the coupling is analytic in the rest of the Q^2 -plane (except on the semiaxis $(-\infty, 0]$) when this approach is applied to a considered observable $\mathcal{D}(Q_0^2)$ carefully at each (complex) Q_0^2 value. On the other hand, when it is applied to $\mathcal{D}(Q_0^2)$ at a fixed chosen Q_0^2 , the resulting PMS coupling $a(Q^2)$ in general has Landau singularities inside the Q^2 plane.

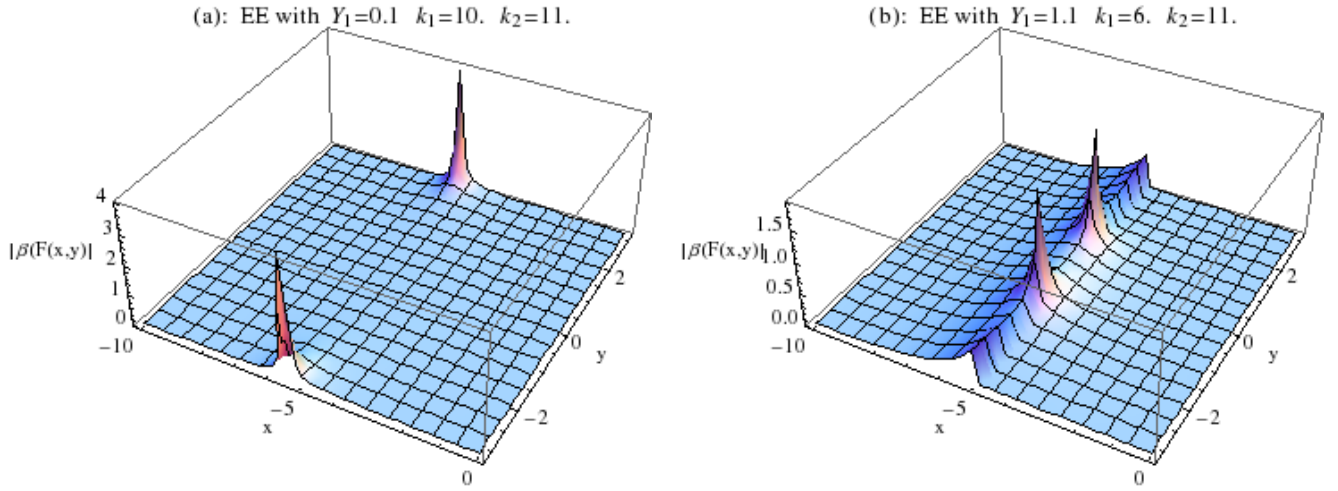


FIG. 3: (a) $|\beta(F(z))|$ as a function of $z = x + iy$, where β has the form (8) with $f(Y)$ having the “EE” form (12) with the values of free parameters Y_1, k_1, k_2 as indicated; (b) same as in (a), but with different values of parameters Y_1 and k_1 .

TABLE I: The four terms (and their sum) in truncated analytic expansions for r_τ , with LB-contributions resummed and the three bLB terms organized in contour integrals of \tilde{a}_{n+1} [cf. Eqs. (A19)-(A20), with Eq. (A17) and the definition (A6)]. RScl parameter is $\kappa = 1$ (i.e., $\mu^2 = Q^2$ on the contour). Included are beta function coefficients c_j ($j = 2, 3, 4$), $a_{\text{in}} \equiv a((3m_c)^2)$ and $a_0 \equiv a(0)$ values, and the threshold mass value M_{thr} (in GeV).

f	r_τ : LB	NLB	N ² LB	N ³ LB	sum	c_2	c_3	c_4	a_{in}	a_0	M_{thr} [GeV]
P[1/0]	0.1122	0.0006	0.0137	0.0007	0.1272	-37.02	0	0	0.0600	0.1901	0.189
P[1/1]	0.1130	0.0006	0.0144	0.0005	0.1285	-37.54	18.84	-9.46	0.0600	0.1992	0.179
EE	0.1364	0.0009	0.0025	0.0048	0.1445	-10.80	-157.62	-644.32	0.0649	0.2360	0.248

In Figs. 4(a), (b) we present beta function $\beta(a)$ as a function of (positive) a , and in Figs. 5(a), (b) the discontinuity function $\rho_1(\sigma) \equiv \text{Im}a(-\sigma - i\epsilon)$ as a function of $\ln(\sigma/Q_{\text{in}}^2)$, for two cases of holomorphic $a(Q^2)$: for the case P[1/0] (i.e., P[2/0] with $r_2 = 0$), and for the case EE ($Y_1 = 0.1; k_1 = 10; k_2 = 11$), respectively. We note that the discontinuity function in both cases shows a clear sudden threshold jump, i.e., the cut starts at $Q^2 = -M_{\text{thr}}^2$, where $M_{\text{thr}} = 0.189$ GeV and 0.248 GeV, respectively.

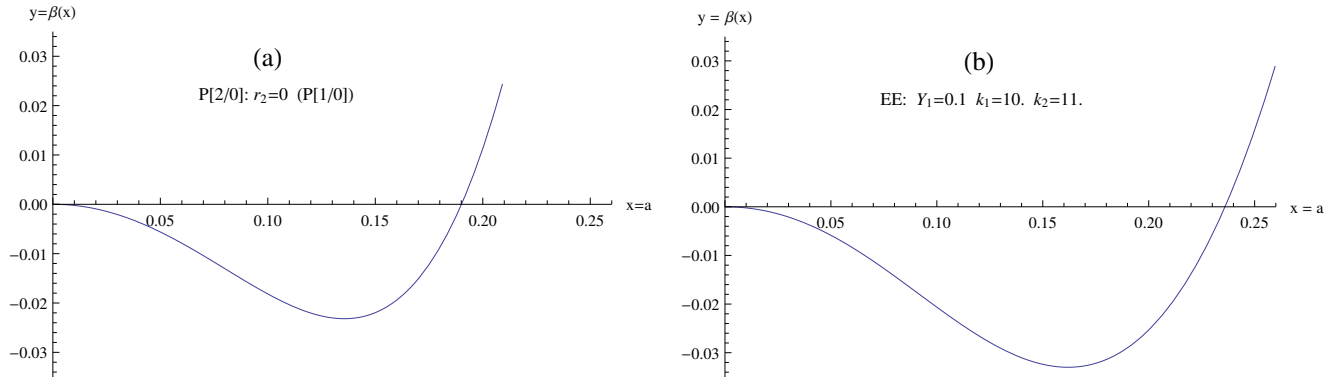


FIG. 4: Beta function $\beta(a)$ as a function of positive a : (a) for the case when $f(Y)$ has the form (10) with $r_2 = 0$, i.e., linear polynomial; (b) for the case when $f(Y)$ is the exponential-related “EE” function (12) with $Y_1 = 0.1; k_1 = 10; k_2 = 11$.

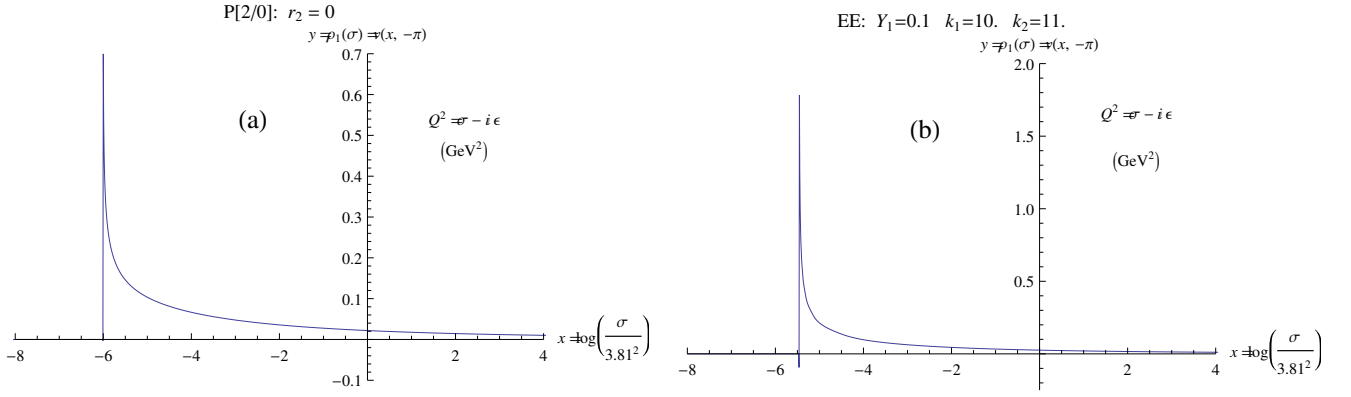


FIG. 5: The discontinuity function $\rho_1(\sigma) = \text{Im } a(Q^2 = -\sigma - i\epsilon) = \text{Im } F(z = x - i\pi) = v(x, y = -\pi)$ as a function of $x = \text{Re}(z) = \ln(\sigma/Q_{\text{in}}^2)$, for the two cases (10) and (12).

Nonetheless, the r_τ values, which are calculated as a leading- β_0 (LB) resummation plus beyond-the-leading- β_0 terms (bLB: NLB $\sim \tilde{a}_2 \sim a^2$, N²LB $\sim \tilde{a}_3 \sim a^3$, N³LB $\sim \tilde{a}_4 \sim a^4$),⁴ are far too low in the described cases, see Table I. We recall that the free parameters in the three aforementioned cases (P[1/0], P[1/1], EE) were chosen in such a way as to make r_τ as big as possible while simultaneously preserving the holomorphic property of $a(Q^2)$. The EE Ansatz gives us the highest $r_\tau \approx 0.145$, but still a long way from the experimental value $r_\tau = 0.203 \pm 0.004$. In the Table I we also display the first few scheme coefficients c_j ($j = 2, 3, 4$), the value of $a_0 \equiv a(0)$, and the threshold mass M_{thr} (in GeV).⁵

IV. MODIFIED BETA FUNCTIONS AND RESULTS

In order to achieve the correct value $r_\tau = 0.203$, and at the same time preserve the holomorphic behavior of $F(z) \equiv a(Q^2)$, we follow in principle the same line of reasoning as in Sec. 3 of Ref. [94] and Sec. IV of Ref. [95]. The idea is to replace in the beta function (8)

$$f(Y) \mapsto f(Y)f_{\text{fact}}(Y). \quad (13)$$

Here, $f_{\text{fact}}(Y)$ should be close to unity for the relevant values $Y \equiv a/a_0 \equiv F(z)/a_0$, i.e., for the values Y around the interval $(0, 1)$ in the complex Y -plane, in order to obtain similar results for $F(z)$ as in the case without an additional factor $f_{\text{fact}}(Y)$. This way there is a high probability that the coupling $F(z)$ for the replaced (modified) beta function remains holomorphic (in the entire open stripe $-\pi < \text{Im} z < +\pi$). We note that now the conditions (9) are applied to $f_{\text{new}}(Y) \equiv f(Y)f_{\text{fact}}(Y)$.

One of the consequences of this condition is that the LB part of r_τ [cf. Eq. (A17)] does not change much by this replacement, and neither do the contour integrals $I(\tilde{a}_{n+1})$ of the NⁿLB contribution [cf. Eqs. (A19)-(A20) and (A6)]. The coefficient $T_1 = 1/12$ of the NLB contribution remains scheme independent (and small), and therefore also the NLB contribution does not change much (and remains small) under the mentioned modification. In Table I we can see that for the original schemes $(c_2^{(0)}, c_3^{(0)}, \dots)$, as defined by the beta functions Eqs. (8) and (10)-(12) [cf. also Eq. (2)], the N²LB and N³LB contributions are too small for achieving the correct value $r_\tau = 0.203$. However, the coefficient T_2 of the N²LB contribution depends strongly on the leading scheme parameter c_2 , it changes linearly with c_2 scheme coefficient of the beta function [cf. Eq. (2)]:⁶ $T_2(c_2) = T_2(c_2^{(0)}) - (c_2 - c_2^{(0)})$. The idea is then to introduce in the beta

⁴ We refer to Appendix A and Refs. [94, 95] for details of calculation of r_τ ; and to Appendix B for the evaluation of the expansion coefficients in the perturbation expansion of the underlying spacelike quantity in the general renormalization schemes.

⁵ These values differ slightly from the corresponding values in Tables II and III of Ref. [95], because here we use for the RGE-running from $Q^2 = M_Z^2$ to $Q^2 = Q_{\text{in}}^2 (= (3m_c)^2)$ the 4-loop truncated (polynomial) beta $\overline{\text{MS}}$ function (in Refs. [94, 95] it was the corresponding Padé P[2/3](a) function), and now we use the world average value $\alpha_s(M_Z^2, \overline{\text{MS}}) = 0.1184$ [101] (in Refs. [94, 95] the value 0.1190 was taken).

⁶ This c_2 -dependence of T_2 can be inferred from relations given in Appendices A and B: Eqs. (A9) give us relations between the d_n and \tilde{d}_n coefficients of the expansions (A5) of the (spacelike) Adler function $d(Q^2)$; Eq. (A20a) relates these coefficients with the coefficients T_n of the LB+bLB expansion (A19) of r_τ , and Eq. (B6b) gives us the c_2 -dependence of d_2 (and thus of T_2), where in Appendix B the Adler function $d(Q^2)$ is a special case of function $\mathcal{F}(Q^2)$ with $\nu_0 = 1$ (and $\mathcal{F}_n = d_n$).

function such $f_{\text{fact}}(Y)$ which fulfills simultaneously the following two conditions:

- (a) $f_{\text{fact}}(Y) \approx 1$ in the sector of the Y complex plane around the $(0, 1)$ interval;
- (b) it decreases the value of c_2 to significantly lower values (from $c_2^{(0)} \sim -10^1$ of Table I to $c_2 \sim -10^2$).

The latter condition increases the T_2 coefficient and the N²LO by about one order of magnitude and thus allows us to obtain the correct value $r_\tau = 0.203$.

In Refs. [94, 95], the $f_{\text{fact}}(Y)$ functions which fulfilled the two mentioned conditions were chosen essentially in the following form:

$$f_{\text{fact}}(Y) = 1 - \frac{K}{B} \left(\frac{BY^2}{1 + BY^2} \right), \quad (14)$$

where $K \sim 10^1$ was needed to obtain $c_2 \sim -10^2$ and thus the correct $r_\tau = 0.203$, and $B \gg K$ ($B \sim 10^3$) was needed to keep $f_{\text{fact}}(Y) \approx 1$ in the Y complex plane around the $(0, 1)$ interval. In this way, the sum of the first four terms, i.e., LB and NⁿLB ($n = 1, 2, 3$) contributions whose coefficients T_n are exactly known, gave the correct value $r_\tau = 0.203$. However, the next term (N⁴LB) was then uncontrollably large: $|r_\tau(\text{N}^4\text{LB})| \propto T_4 \sim c_4 \sim BK \sim 10^6\text{-}10^7$. This took place due to the fact that the expansion of $f_{\text{fact}}(Y)$ in powers of Y^2 has a huge coefficient BK at Y^4 ($(a/a_0)^4$)

$$f_{\text{fact}}(Y)_{\text{exp}} = 1 - KY^2 + (BK)Y^4 - (B^2K)Y^6 + \dots \quad (15)$$

Within the present work we try to avoid this unwanted behavior in the following way: we modify the expression (14) for f_{fact} into an expression $\mathcal{F}_{\text{fact}}$ such that the offending coefficients in the expansion (15) disappear

$$f_{\text{fact}}(Y) \mapsto \mathcal{F}_{\text{fact}}(Y), \quad (16a)$$

$$\text{such that } f_{\text{fact}}(Y)_{\text{exp}} \mapsto \mathcal{F}_{\text{fact}}(Y)_{\text{exp}} = 1 - KY^2 + \mathcal{O}((Y^2)^{P+1}), \quad (16b)$$

where the subscript “exp” denotes the expanded form of the corresponding function, P is a large chosen integer, $K \sim 10^1$ as required by the mentioned condition (b), and at the same time requiring that the condition (a) survives: $\mathcal{F}_{\text{fact}}(Y) \approx 1$ in the sector of the Y complex plane around the $(0, 1)$ interval, in order to preserve analyticity of $F(z)$. It turns out that such a transformation is possible [Eqs. (21)-(22)], as we show in the following. Let $g(\omega)$ be a function whose expansion around $\omega = 0$ is

$$g(\omega)_{\text{exp}} = \mathcal{C}_1\omega + \mathcal{C}_2\omega^2 + \dots \quad (17)$$

Consider the finite group of rotations of the complex plane given by

$$\omega \mapsto \exp\left(i\frac{2\pi}{P}k\right)\omega \quad \text{for } k = 0, \dots, P-1.$$

Now consider the average of the images of $h(\omega) \equiv g(\omega)/\omega$ under this group of rotations of order P . We denote this average by $\tilde{h}_P(\omega) \equiv \tilde{g}_P(\omega)/\omega$:

$$\frac{\tilde{g}_P(\omega)}{\omega} \equiv \tilde{h}_P(\omega) = \frac{1}{P} \sum_{k=0}^{P-1} h(\exp\left(i\frac{2\pi}{P}k\right)\omega). \quad (18)$$

It is straightforward to verify that all the terms with exponents that are not divisible by P are annihilated.⁷ We are left with

$$\tilde{g}_P(\omega)_{\text{exp}} = \mathcal{C}_1\omega + \mathcal{C}_{P+1}\omega^{P+1} + \mathcal{C}_{2P+1}\omega^{2P+1} \dots, \quad (19)$$

⁷ Using the expansion (17) in Eq. (18) leads to $\tilde{h}_P(\omega)_{\text{exp}} = (1/P) \sum_{k=0}^{P-1} \sum_{N=0}^{\infty} C_{N+1} \exp(i2\pi kN/P) \omega^N = \sum_{N=0}^{\infty} C_{N+1} \omega^N \times (1/P) \sum_{k=0}^{P-1} \exp(i2\pi kN/P)$. When $N \neq 0, P, 2P, \dots$, we have $\sum_{k=0}^{P-1} \exp(i2\pi kN/P) = (\exp(i2\pi N) - 1) / (\exp(i2\pi N/P) - 1) = 0$; and when $N = 0, P, 2P, \dots$, we have $\sum_{k=0}^{P-1} \exp(i2\pi kN/P) = P$. This then leads to the expression (19).

TABLE II: Same as in Table I, but now beta function is changed by the substitution $f(Y) \mapsto f(Y)\mathcal{F}_{\text{fact}}(Y; N)$, Eqs. (21)-(23), where $N = 0$ and $N = 25$. Included are new beta function coefficients c_j ($j = 2, 3, 4$), a_{in} and a_0 values, and the threshold mass value M_{thr} (in GeV).

f	N	B	K	r_τ : LB	NLB	N ² LB	N ³ LB	sum	c_2	c_3	c_4	a_{in}	a_0	M_{thr} [GeV]
P[1/0]	0	5000	6.8	0.1055	0.0006	0.0911	0.0058	0.2029	-224.67	-333.73	2.59×10^7	0.0574	0.1903	0.159
P[1/0]	25	2100	6.8	0.1054	0.0006	0.0913	0.0058	0.2030	-225.19	-334.52	6966.7	0.0574	0.1901	0.159
P[1/1]	0	4650	7.3	0.1056	0.0006	0.0910	0.0058	0.2030	-220.89	-307.36	2.14×10^7	0.0573	0.1994	0.149
P[1/1]	25	2000	7.3	0.1055	0.0006	0.0914	0.0059	0.2032	-221.49	-308.18	6895.3	0.0573	0.1992	0.149
EE	0	1150	5.6	0.1235	0.0007	0.0690	0.0098	0.2029	-110.45	-333.35	2.04×10^6	0.0608	0.2369	0.196
EE	25	500	5.6	0.1231	0.0007	0.0694	0.0098	0.2029	-111.32	-336.33	441.47	0.0606	0.2360	0.196

i.e., the expansion series of $\tilde{g}_P(\omega)$ has vanishing lowest-order terms (with the exception of the linear one). When applying this approach to $g(\omega) \equiv f_{\text{fact}}(Y) - 1$ of Eq. (14), with $\omega = Y^2$, we obtain

$$\mathcal{F}_{\text{fact}}(Y) = 1 - \frac{K}{B} \left(\frac{BY^2}{1 - (-BY^2)^P} \right), \quad (20)$$

It turns out that the condition (a) is fulfilled only when P is odd: $P = 2N + 1$ (because for $P = 2N$ beta function has a pole at small positive $Y = 1/\sqrt{B}$). Consequently, we use $P = 2N + 1$

$$\mathcal{F}_{\text{fact}}(Y; N) = 1 - \frac{K}{B} \left(\frac{BY^2}{1 + (BY^2)^{2N+1}} \right) \quad (21a)$$

$$= \frac{1 + (BY^2)^{2N+1} - KY^2}{1 + (BY^2)^{2N+1}}, \quad (21b)$$

and the expansion around $Y = 0$ is

$$\Rightarrow \mathcal{F}_{\text{fact}}(Y; N)_{\text{exp}} = 1 - KY^2 + \frac{K}{B} [(BY^2)^{2N+2} - (BY^2)^{4N+3} + (BY^2)^{6N+4} \dots] . \quad (22)$$

We recall that $K \sim 10^1$ and $B \gg K$. The new considered beta functions are now, according to Eqs. (8)-(9), (13), (16)

$$\beta(F) = -\beta_0 F^2 (1 - Y) f(Y) \mathcal{F}_{\text{fact}}(Y; N), \quad (23)$$

where $F \equiv a(Q^2)$, $Y \equiv F/a_0 \equiv a/a_0$. For $f(Y)$ we stick to the original options Eqs. (10) ($r_2 = 0$) or (11) or (12). We note that the conditions (9) are applied now to $f_{\text{new}}(Y) \equiv f(Y)\mathcal{F}_{\text{fact}}(Y)$.

Here we recall once more that the physical condition of obtaining large enough value of r_τ (≈ 0.203) imposed on us: (1) the condition of having a large value $K \sim 10^1$ in the expansion $\mathcal{F}_{\text{fact}}(Y; N)_{\text{exp}} = 1 - KY^2 + \mathcal{O}(Y^3)$ with the coefficients at higher powers of Y under control; (2) and simultaneously the condition $\mathcal{F}_{\text{fact}}(Y; N) \approx 1$ in the sector of the Y complex plane around the $(0, 1)$ interval. Mathematically, these two conditions tend to be in general in conflict, which explains why it was so difficult to obtain a solution, such as Eqs. (21)-(22), reconciling both of them.⁸

In Refs. [94, 95] only the modification with $\mathcal{F}_{\text{fact}}(Y; N = 0)$ was investigated. We now perform the numerical integration of the RGE (5) for many different N , as high as $N = 25$, and adjust B ($\sim 10^3$) and K ($\sim 10^1$) so that the correct value of r_τ is obtained by adding the first four terms. The higher terms $r_\tau(N^n \text{LB})$ ($n \geq 4$) are now under control, they are estimated to contribute less than 0.001. And the holomorphic behavior of $F(z) \equiv a(Q^2)$ is preserved when N increases. In Table II we present the results analogous to those in Table I, but now with $f(Y) \mapsto f(Y)\mathcal{F}_{\text{fact}}(Y; N)$ for the choices $N = 0$ and $N = 25$, again for the three cases of $f(Y)$: Eq. (10) with $r_2 = 0$ (P[1/0]); Eq. (11) with $u_1 = -0.1$ (P[1/1]); Eq. (12) with $Y_1 = 0.1$, $k_1 = 10.0$, $k_2 = 11.0$. Note how strongly c_4

⁸ The problem of too low value of r_τ was already encountered and criticized in Ref. [97] in the case of the analytic (and nonperturbative) QCD model APT of Refs. [48–52]. The formal reason for the problem of (the tendency to) too low r_τ was identified in Ref. [97] in the fact that in analytic models the Landau cut of the coupling at positive $Q^2 \equiv -\sigma$ ($0 < Q^2 \leq \Lambda_{L-}^2$) is missing and therefore the integral for r_τ with positive integrand, e.g. Eq. (A17), has a “missing” integration interval $-\Lambda_{L-}^2 \leq \sigma < 0$ [$\sigma \equiv te^{\bar{c}} m_\tau^2$ in Eq. (A17)] along the Landau cut and thus gives in general too small r_τ .

gets suppressed when going from $N = 0$ to $N = 25$. There is some freedom of varying K and B so that $r_\tau = 0.203$ is obtained. However, when we decrease B (at a given N), K has to be increased somewhat, and the convergence properties of the first four terms in the sum for r_τ deteriorate: LB contribution decreases, and the N^2 LB contribution increases. Since we want to have the first (LB) term clearly dominant, we are forced to use relatively high values of B ($\sim 10^3$). Though, when N increases, we can use somewhat lower values of B while still maintaining the same convergence quality.

It turns out that Figures 2(a), 3(a), 4 and 5 change only a little when the modifications with the parameters (N, B, K) as given in Table II are performed. Most importantly, the holomorphic behavior of $F(z) \equiv a(Q^2)$, as signalled by Figs. 2(a) and 3(a) in the nonmodified case ($K = 0$), is preserved.⁹ For this, we present in Figs. 6 the

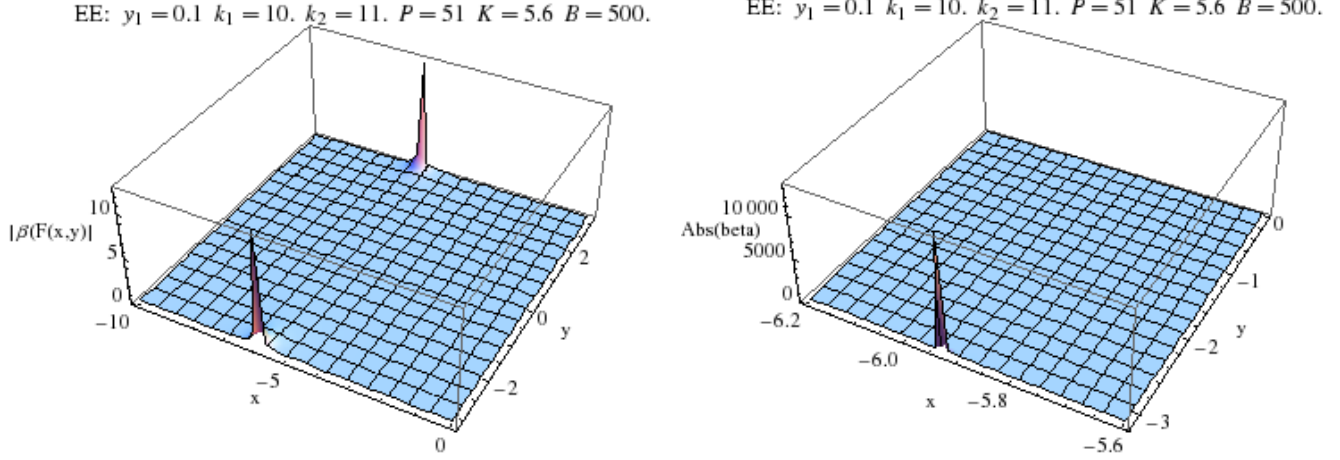


FIG. 6: $|\beta(F(z))|$ as a function of $z \equiv \ln(Q^2/Q_{\text{in}}^2) = x + iy$, as in Fig. 3(a), but now with the modified beta function with $P = 51$ ($= 2N + 1$), $K = 5.6$ and $B = 500$; the right-hand figure has more details around the pole. No Landau singularities (i.e., singularities inside the z stripe) appear.

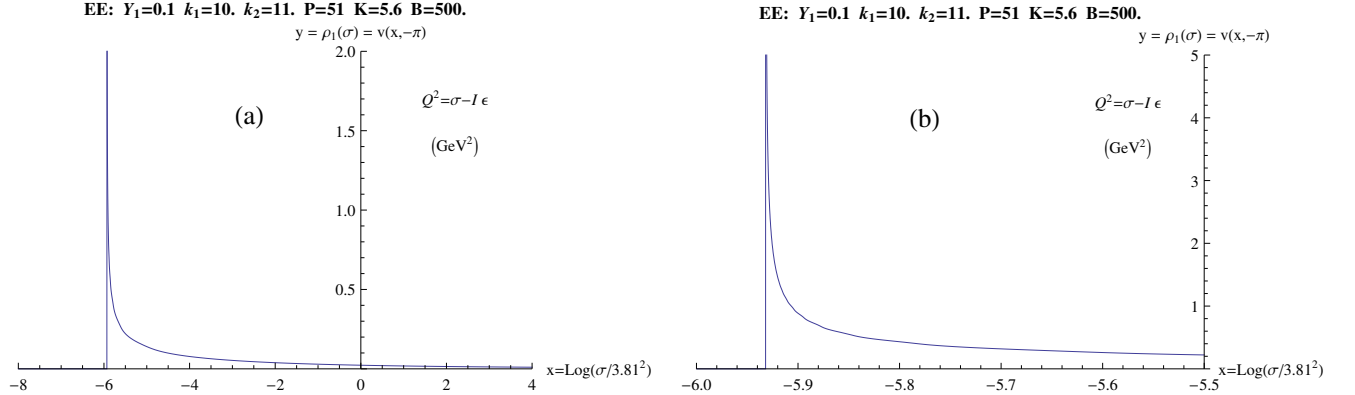


FIG. 7: (a) The discontinuity function $\rho_1(\sigma) = \text{Im } a(Q^2 = -\sigma - i\epsilon) = \text{Im } F(z = x - i\pi) = v(x, y = -\pi)$ as a function of $x = \text{Re}(z) = \ln(\sigma/Q_{\text{in}}^2)$, for the EE case with modified beta function ($P = 51$, i.e., $N = 25$); (b) enlarged picture around the threshold.

behavior of the function $|\beta(F(z))|$ for the EE case with $N = 25$ (i.e., $P \equiv 2N + 1 = 51$), which clearly indicates that there are no Landau singularities. In addition, in Figs. 7 we present for this case the discontinuity function $\rho_1(\sigma) \equiv \text{Im } a(-\sigma - i\epsilon)$ as a function of $\ln(\sigma/Q_{\text{in}}^2)$, and in Figs. 8 the running coupling $a(Q^2)$, and its logarithmic derivatives $\tilde{a}_2(Q^2)$ and $\tilde{a}_3(Q^2)$ [defined in Eq. (A6)], for positive Q^2 .

While the considered perturbative beta functions modified by the factor (21) give us simultaneously the correct value of r_τ and perturbative holomorphic coupling $a(Q^2)$, one may worry that the introduction of the large coefficients \sim

⁹ This is true also in the other considered case: $f(Y) = P[1/1](Y)$ with $u_1 = -1$.

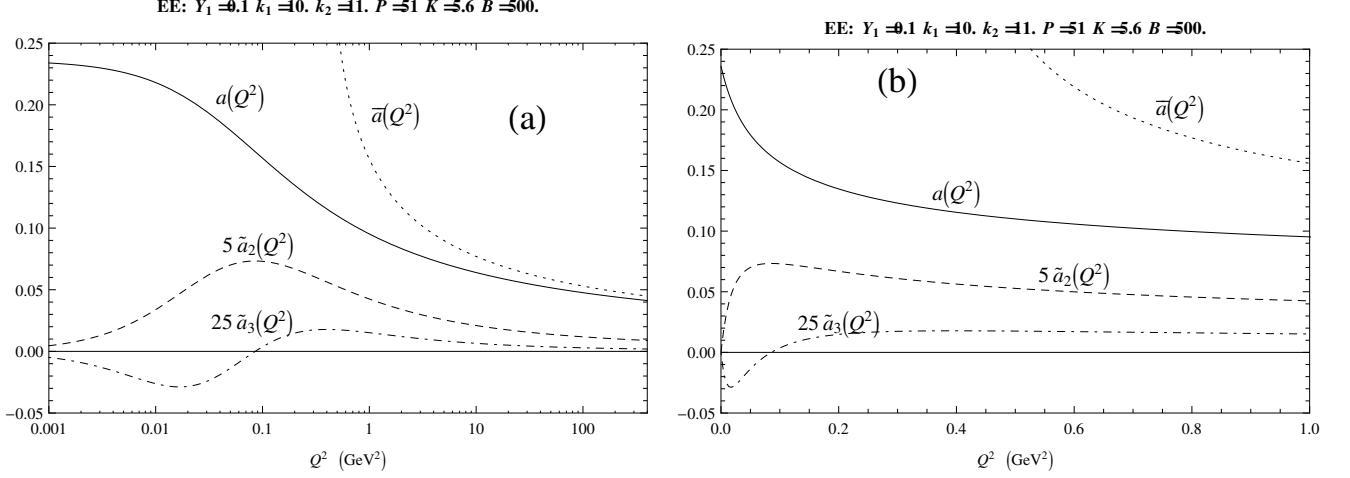


FIG. 8: (a) The running coupling $a(Q^2)$ as a function of positive Q^2 , for the EE case with modified beta function ($P = 51$, i.e., $N = 25$), when Q^2 is on a logarithmic scale; included are the logarithmic derivatives $\tilde{a}_2(Q^2)$ and $\tilde{a}_3(Q^2)$, Eq. (A6), rescaled by factors 5 and 25 for better visibility; for comparison, the corresponding \overline{MS} coupling $\bar{a}(Q^2)$ is included; (b) same as in Fig. (a), but on linear Q^2 scale for low positive Q^2 .

B^{2N+1} (where $B \sim 10^3$) in the beta function [cf. Eq. (22)] represents an anomalous mass independent renormalization scheme, in the sense that the growth of the coefficients of beta function at large order becomes responsible for a growth of the coefficients of the physical spacelike physical quantities which is faster than the growth coming from the leading (UV or IR) renormalon. Here we will argue that if $P \equiv 2N + 1$ in (21) is large enough, e.g. $P \geq 51$, then the renormalon growth of the coefficients will dominate over the growth from the beta function coefficients. For example, in the case of the timelike quantity r_τ , the underlying spacelike quantity is Adler function $d(Q^2)$ (with $N_f = 3$), cf. Eq. (A4).

Let us consider a general spacelike physical quantity $\mathcal{D}(Q^2)$, whose expansion is

$$\mathcal{D}(Q^2) = a(Q^2) + d_1 a(Q^2)^2 + \dots + d_n a(Q^2)^{n+1} + \dots \quad (24)$$

The expansion of its Borel transform is

$$B_{\mathcal{D}}(b) = 1 + \frac{d_1}{1!\beta_0} b + \dots + \frac{d_n}{n!\beta_0^n} b^n + \dots \quad (25)$$

It turns out that this function can have (renormalon) poles only at nonzero integer values $b = \pm 1, \pm 2, \dots$, cf. Ref. [113]. The closer the renormalon pole is to the origin, the faster is the increase of the coefficients d_n with n . Let us assume that the pole is at either $b = 1$ or $b = -1$. Then, in the large- β_0 approximation, the coefficients d_n behave at large n as

$$|d_n| \approx \mathcal{K} n! \beta_0^n \sim n! \beta_0^n, \quad (26)$$

where $\mathcal{K} \sim 1$.¹⁰ We recall that in our notation, $a \equiv \alpha_s/\pi$ and $\beta_0 = (1/4)(11 - 2N_f/3)$, i.e., $\beta_0 = 9/4$ when $N_f = 3$. On the other hand, the perturbative scheme independence of the physical quantity $\mathcal{D}(Q^2)$ implies that the coefficient d_n has a specific dependence on the scheme coefficients c_2, c_3, \dots, c_n of the beta function expansion (2). In particular, the dependence on c_n is

$$d_n(c_n, c_{n-1}, \dots, c_2) = -\frac{1}{n-1} c_n + f_n(c_{n-1}, \dots, c_2). \quad (27)$$

On the other hand, it is straightforward to check that for the beta function (23) modified by the factor (21), with $B \sim 10^3$, $K \sim 10^1$ and $P = 2N + 1$, the first anomalous (large) beta coefficient is

$$c_{2P+2} \approx (-1)^{P+1} \frac{K}{a_0^2} \left(\frac{B}{a_0^2} \right)^P. \quad (28)$$

¹⁰ We recall that the renormalon problem is reflected in this growth of the coefficients d_n , and is not related with the existence or nonexistence of the Landau singularities of the running coupling $a(Q^2)$. The question whether the Landau singularities appear or not is a problem of the running coupling and its beta function only, cf. comments at the end of Sec. 2.2 of the review Ref. [113].

This implies, together with Eq. (27), that the first d_n with anomalously large beta coefficient is

$$d_{2P+2} = \frac{(-1)^P K}{2P+1} \left(\frac{B}{a_0^2} \right)^P + \dots, \quad (29)$$

where the dots stand for a contribution that is independent of the anomalous scheme parameter B . We require that the contribution (29) of the anomalous scheme to the coefficient d_{2P+2} is less than the contribution of the $b = \pm 1$ (leading) renormalon (26), and this implies

$$\begin{aligned} \frac{1}{2P+1} \frac{K}{a_0^2} \left(\frac{B}{a_0^2} \right)^P &< (2P+2)! \beta_0^{2P+2} \\ \Rightarrow \text{rat}(P) &\equiv \frac{KB^P}{a_0^{2P+2} (2P+1)(2P+2)! \beta_0^{2P+2}} < 1 \end{aligned} \quad (30)$$

In the considered case of EE with $N = 25$ ($P = 51$), i.e., the last line of Table II, it turns out that this ratio $\text{rat}(P)$ is 0.09, i.e., the $b = \pm 1$ renormalon growth of the coefficients d_n clearly dominates over the growth of d_n coming from the scheme. For the other two cases (P[1/1] and P[1/0]) in Table II, this ratio is still huge, primarily due to the larger value of B (≈ 2000), and a significantly larger value of P is needed.¹¹ If the renormalon effects are accounted for beyond the large- β_0 approximation, the growth of the coefficients d_n becomes even slightly faster [113]. Furthermore, in the specific case of r_τ , where the underlying coefficients d_n are those of Adler function, the leading renormalon is at $b = -1$ and it is double (quadratic), so that the $|d_n|$ coefficients grow even slightly faster than in Eq. (26), as $\sim (n+1)! \beta_0^n$.

V. BOREL SUM RULES IN V+A CHANNEL OF TAU LEPTON SEMIHADRONIC DECAYS

In this Section we extract the four and six-dimensional condensates $\langle O_4^{(V+A)} \rangle = (1/6) \langle a G_{\mu\nu}^\alpha G_{\mu\nu}^\alpha \rangle$ and $\langle O_6^{(V+A)} \rangle$ appearing in the Operator product expansion (OPE) of the $V + A$ quark current correlator $\Pi(Q^2)$, based on the measurements of the τ -lepton semihadronic decays. We use the Borel transform sum rules, following closely the approach of Ref. [88], where the evaluation was performed for the (nonperturbative) 2-delta analytic QCD (2 δ anQCD) model of Ref. [74]. That approach followed the Borel transform sum rule methods of Refs. [97, 114]. We outline only the main features of the approach and refer for details of the approach to Ref. [88] and Ref. [114].

The starting point is the identity (sum rule)

$$\int_0^{\sigma_0} d\sigma g(-\sigma) \omega_{\text{exp}}(\sigma) = -i\pi \oint_{|Q^2|=\sigma_0} dQ^2 g(Q^2) \Pi_{\text{th}}(Q^2), \quad (31)$$

where $g(Q^2)$ is an analytic (holomorphic) function in the entire Q^2 complex plane, which characterizes the specific sum rule. The contour integration on the right-hand side is in the counterclockwise direction, and $\omega(\sigma)$ is the spectral function of the $V + A$ quark current correlator function $\Pi(Q^2)$

$$\omega(\sigma) \equiv 2\pi \text{Im} \Pi(Q^2 = -\sigma - i\epsilon). \quad (32)$$

The identity (31) comes from applying the Cauchy theorem to the function $g(Q^2)\Pi(Q^2)$ and accounting for the correct holomorphic behavior of the correlator $\Pi(Q^2)$ as required by the general principles of quantum field theories. The same type of holomorphic behavior is respected by the QCD running couplings $a(Q^2)$ and $\tilde{a}_n(Q^2)$ in the schemes considered in this work. Therefore, the theoretically evaluated correlators $\Pi_{\text{th}}(Q^2) [\Leftrightarrow \mathcal{D}_{\text{Adl}}(Q^2)]$, at each order of truncation in the considered holomorphic schemes, have the analytic behavior consistent with the identity (31).

In the present case, we are interested in $V+A$ channel of τ lepton semihadronic nonstrange decays. The experimental spectral function $\omega_{\text{exp}}(\sigma)$ on the left-hand side of the sum rule (31) is obtained from the invariant-mass spectra of the τ lepton strangeless decays with the squared invariant mass σ in the interval $0 < \sigma < \sigma_0$. Our analysis here is based

¹¹ If we use in the numerical integration of the RGE a very large value of P ($P > 100$), the calculation (with MATHEMATICA) becomes either very time-consuming, or it does not perform due to overflow problems.

on the data of ALEPH Collaboration [115–117]. On the right-hand side of the sum rule is the correlator function $\Pi(Q^2)$, which is theoretically evaluated with OPE

$$\Pi(Q^2) = -\frac{1}{2\pi^2} \ln(Q^2/\mu^2) + \Pi(Q^2; D=0) + \sum_{n \geq 2} \frac{\langle O_{2n} \rangle}{(Q^2)^n} (1 + \mathcal{C}_n a(Q^2)) . \quad (33)$$

Note that $n = D/2$ where D denotes the operator dimension of the local operators contributing to the OPE of $\Pi(Q^2)$. The $D = 2$ ($n = 1$) term is proportional to the current masses of u and d quarks, and is negligible. For us the relevant terms are $D = 4, 6$ ($n = 2, 3$). Further, it can be checked that the terms proportional to $\mathcal{C}_n a(Q^2)$ will be negligible in the Borel sum rules applied here (cf. footnote 20 of Ref. [88]).

For the evaluation of the right-hand side of the sum rule (31), it turns out convenient to integrate it by parts

$$\int_0^{\sigma_0} d\sigma g(-\sigma) \omega_{\text{exp}}(\sigma) = -\frac{i}{2\pi} \int_{\phi=-\pi}^{\pi} \frac{dQ^2}{Q^2} \mathcal{D}_{\text{Adl}}(Q^2) [\mathcal{G}(Q^2) - \mathcal{G}(-\sigma_0)] \big|_{Q^2=\sigma_0 \exp(i\phi)} , \quad (34)$$

where \mathcal{G} is any function satisfying

$$\frac{d\mathcal{G}(Q^2)}{dQ^2} = g(Q^2) , \quad (35)$$

and $\mathcal{D}_{\text{Adl}}(Q^2)$ is the full massless Adler function

$$\mathcal{D}_{\text{Adl}}(Q^2) \equiv -2\pi^2 \frac{d\Pi(Q^2)}{d \ln Q^2} \quad (36a)$$

$$= 1 + d(Q^2) + 2\pi^2 \sum_{n \geq 2} \frac{n \langle O_{2n} \rangle}{(Q^2)^n} , \quad (36b)$$

where the terms with $\mathcal{C}_n a(Q^2)$ were neglected, as mentioned earlier. The dimension $D = 0$ part of the correlator is directly related to the (strangeless and massless) canonical Adler function $d(Q^2)$ of Appendix A [Eqs. (A5), (A11)–(A14)]

$$d(Q^2) = -2\pi^2 \frac{d\Pi(Q^2; D=0)}{d \ln Q^2} . \quad (37)$$

In the sum rule (31), the analytic function $g(Q^2)$ is usually taken to be either an exponential function $\propto \exp(Q^2/M^2)$ (Borel sum rules, Refs. [114, 118]), or a Gaussian function $\propto \exp((Q^2/M^2)^2)$ (Gaussian sum rules, Ref. [118]), or powers $\propto (Q^2)^N$. The integrals of the latter approach are called moments, and the corresponding sum rules are usually called finite energy sum rules, cf. Refs. [118–126].

Although the approach with moments is more widely used in the literature, we will calculate the Borel transforms, i.e., we will apply the Borel sum rules [114], the main reason being that we already have experience and acquired confidence in such calculations, cf. Ref. [88]. Nonetheless, it would be interesting to apply in the future the moment approach to the considered holomorphic schemes.

Therefore, our choice for $g(Q^2)$ here will be

$$g(Q^2) = \frac{1}{M^2} \exp(Q^2/M^2) , \quad \mathcal{G}(Q^2) = \exp(Q^2/M^2) , \quad (38)$$

where M^2 are chosen complex scales with $\text{Re}(M^2) > 0$. The expressions in the sum rules (31) and (34) become Borel transforms $B(M^2)$, and we choose there for the upper integration bound the maximal possible value $\sigma_0 = m_\tau^2$ ($\approx 3.16 \text{ GeV}^2$).¹² The Borel sum rule thus has the form

$$B_{\text{exp}}(M^2) = B_{\text{th}}(M^2) , \quad (39)$$

¹² If σ_0 is taken well below m_τ^2 , the duality-violating effects become important and must be taken into account, see Refs. [127, 128].

where

$$B_{\text{exp}}(M^2) \equiv \int_0^{m_\tau^2} \frac{d\sigma}{M^2} \exp(-\sigma/M^2) \omega_{\text{exp}}(\sigma) , \quad (40a)$$

$$B_{\text{th}}(M^2) \equiv B(M^2; D=0) + 2\pi^2 \sum_{n \geq 2} \frac{\langle O_{2n} \rangle}{(n-1)! (M^2)^n} , \quad (40b)$$

The $D = 0$ part is

$$B(M^2; D=0) = (1 - \exp(-m_\tau^2/M^2)) + \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi d(Q^2 = m_\tau^2 e^{i\phi}) \left[\exp\left(\frac{m_\tau^2 e^{i\phi}}{M^2}\right) - \exp\left(-\frac{m_\tau^2}{M^2}\right) \right] . \quad (41)$$

For small positive $\text{Re}(M^2)$, the Borel transform suppresses strongly the contributions of $\omega_{\text{exp}}(\sigma)$ at high energies (high σ) where the experimental errors are larger. Further, the OPE higher dimension terms are suppressed in the Borel transform by a factor $1/(n-1)!$. In the real part of the Borel transform, OPE term contributions of specific dimension D are eliminated if the complex scales M^2 are chosen along specific rays $M^2 = |M^2| \exp(i\psi)$ in the complex plane. This facilitates the determination of the remaining condensates $\langle O_{2n} \rangle$ by comparing the theoretical expressions with the experimental values $\text{Re}B_{\text{exp}}(|M^2| \exp(i\psi))$. For example, if $\psi = \pi/6, \pi/4$, then $D = 6, 4$ terms do not contribute, respectively, because $\text{Re}(\exp(i\pi/2)) = 0$. Therefore, when ignoring terms with $D > 6$, we have

$$\text{Re}B_{\text{exp}}(|M|^2 e^{i\pi/6}) = \text{Re}B(|M|^2 e^{i\pi/6}; D=0) + \pi^2 \frac{\langle O_4 \rangle}{|M|^4} , \quad (42a)$$

$$\text{Re}B_{\text{exp}}(|M|^2 e^{i\pi/4}) = \text{Re}B(|M|^2 e^{i\pi/4}; D=0) - \pi^2 \frac{\langle O_6 \rangle}{\sqrt{2}|M|^6} , \quad (42b)$$

We note that in the considered $V + A$ channel, the $D = 4$ operator is proportional to the gluon condensate

$$\langle O_4^{(V+A)} \rangle = \frac{1}{6} \langle a G_{\mu\nu}^\alpha G_{\mu\nu}^\alpha \rangle , \quad (43)$$

and the $D = 6$ operator, in the vacuum saturation approximation, is nonnegative and proportional to the square of the quark-antiquark condensate [97, 114]

$$\langle O_6^{(V+A)} \rangle \approx \frac{128\pi^2}{81} a \langle \bar{q}q \rangle^2 . \quad (44)$$

Here, as throughout this work, the notation $a \equiv \alpha_s/\pi$ is used.

The experimental values $\text{Re}B_{\text{exp}}(|M^2| \exp(i\psi))$ we use here are those of Figs. 4 and 5(a),(b) of Ref. [97] (they were used also in Ref. [88]), which are based on the values $\omega_{\text{exp}}(\sigma)$ of the ALEPH 1998 data [115]. For the theoretical values $\text{Re}B_{\text{th}}(|M^2| \exp(i\psi))$, the evaluation of the contour integrals (41) of the canonical Adler function $d(Q^2)$ was performed with renormalization scale $\mu^2 = Q^2$, in the EE renormalization scheme: beta function (23) with $f(Y)$ of Eq. (12) [with $Y_1 = 0.1$; $k_1 = 10$; $k_2 = 11$.] and $\mathcal{F}_{\text{fact}}$ of the form (21) with $B = 500$, $K = 5.6$ and $P \equiv 2N + 1 = 51$. For completeness, we write down here also values of the other beta function parameters Y_2 and $a_0 \equiv \mathcal{A}_1(0)$ in this specific EE scheme, obtained numerically through the conditions (9b)-(9c) applied to $f_{\text{new}}(Y) \equiv f(Y)\mathcal{F}_{\text{fact}}(Y)$: $Y_2 = 0.1839408532$ and $a_0 = 0.2360296246$. We choose this scheme because, as shown hitherto in this work, it represents an analytic (holomorphic) perturbative QCD, gives the correct value of r_τ decay ratio (cf. the last line in Table II) and the growth of the coefficients d_n with rising n is dominated by the leading renormalon $b = \pm 1$ (cf. the previous Section) and not by the beta function.

Furthermore, r_τ was calculated in the LB+bLB approach, which is applicable if the running coupling $a(Q^2)$ is holomorphic and which uses the maximal amount of the presently available information on the perturbation coefficients of Adler function, and is thus considered as one of the most effective resummation approaches for the τ decay physics quantities. Therefore, we apply the LB+bLB approach also in the calculation of the contour integrals (41) for the Borel sum rules. We refer to Appendix C for some formal details of the LB+bLB approach to the Borel sum rules (analogous to Appendix A which explains the calculation of r_τ in LB+bLB approach).

The experimental and the theoretical results are given in Figs. 9(a),(b) for $\psi \equiv \arg(M^2) = \pi/6, \pi/4$, respectively, for the interval $0.68 \text{ GeV}^2 < |M^2| < 1.50 \text{ GeV}^2$. Comparison of the (EE scheme) theoretical curves with the experimental

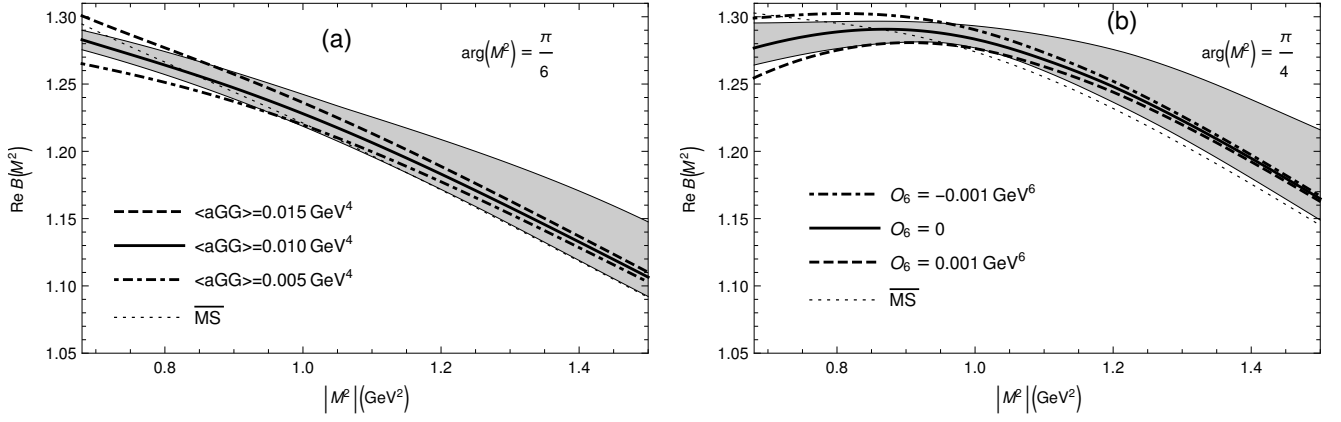


FIG. 9: (a) $\text{Re}B(M^2)$ for $M^2 = |M^2| \exp(i\pi/6)$; (b) for $M^2 = |M^2| \exp(i\pi/4)$. The grey band represents the experimental data. In Fig. (a), the (EE scheme) theoretical curves correspond to $\langle aG_{\mu\nu}^\alpha G_{\mu\nu}^\alpha \rangle = (0.010 \pm 0.005) \text{ GeV}^4$; the $\overline{\text{MS}}$ curve with $\langle aG_{\mu\nu}^\alpha G_{\mu\nu}^\alpha \rangle = 0.0059 \text{ GeV}^4$ is included as the dotted curve. In Fig. (b), the (EE scheme) theoretical curves correspond to $\langle O_6^{(V+A)} \rangle_6 = (0 \pm 0.001) \text{ GeV}^6$; the $\overline{\text{MS}}$ curve with $\langle O_6^{(V+A)} \rangle_6 = -1.8 \times 10^{-3} \text{ GeV}^6$ is included as the dotted curve.

bands allows us to make an “educated guess” estimate of the condensate values

$$\langle aG_{\mu\nu}^\alpha G_{\mu\nu}^\alpha \rangle = (0.010 \pm 0.005) \text{ GeV}^4, \quad (45a)$$

$$\langle O_6^{(V+A)} \rangle = (0 \pm 0.001) \text{ GeV}^6. \quad (45b)$$

This can be compared with the values extracted by the Borel sum rule approach in $\overline{\text{MS}}$ scheme [where truncated series is taken for $d(Q^2)$] in Ref. [88]: $\langle aGG \rangle = (0.0059 \pm 0.0049) \text{ GeV}^4$ and $\langle O_6^{(V+A)} \rangle = (-1.8 \pm 0.9) \times 10^{-3} \text{ GeV}^6$. It is interesting that at $\psi = \pi/4$ we extract (in the EE scheme) the values of $\langle O_6^{(V+A)} \rangle$, Eq. (45b), which are compatible with nonnegative values. This nonnegativity is compatible with the expectation based on the vacuum saturation approximation Eq. (44). On the other hand, the extracted values of $\langle O_6^{(V+A)} \rangle$ in the $\overline{\text{MS}}$ Borel sum rule approach are not compatible with the nonnegativity Eq. (44).

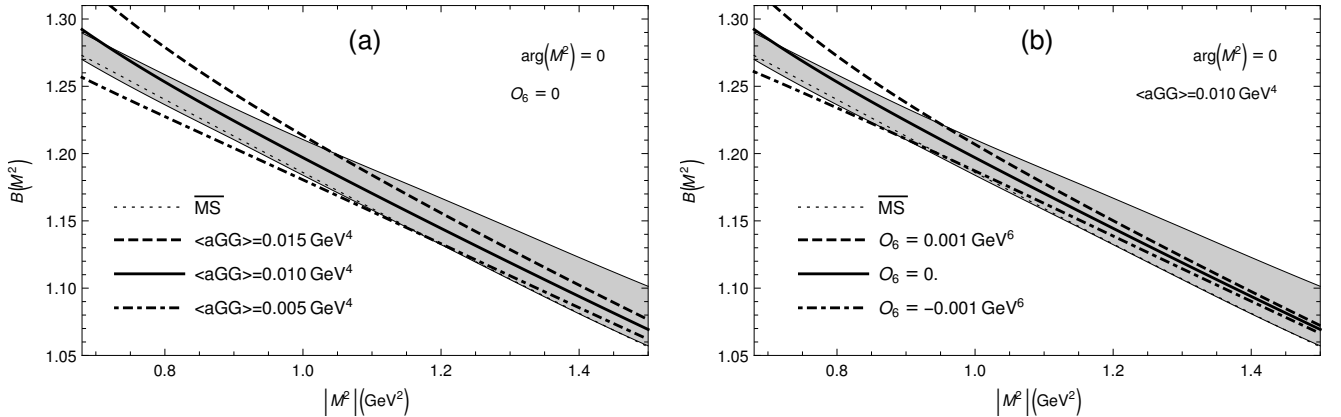


FIG. 10: $B(M^2)$ for $M^2 = |M^2|$ (positive) scales. The grey band represents the experimental data. In Fig. (a), the (EE scheme) theoretical curves correspond to $\langle aG_{\mu\nu}^\alpha G_{\mu\nu}^\alpha \rangle = (0.010 \pm 0.005) \text{ GeV}^4$ and the central value $\langle O_6^{(V+A)} \rangle = 0$. In Fig. (b), the (EE scheme) theoretical curves correspond to $\langle O_6^{(V+A)} \rangle = (0 \pm 0.001) \text{ GeV}^6$ and the central value $\langle aG_{\mu\nu}^\alpha G_{\mu\nu}^\alpha \rangle = 0.010 \text{ GeV}^4$. For comparison, the $\overline{\text{MS}}$ curve is included in both figures as the dotted line, with its own central values $\langle aG_{\mu\nu}^\alpha G_{\mu\nu}^\alpha \rangle = 0.0059 \text{ GeV}^4$ and $\langle O_6^{(V+A)} \rangle = (-1.8) \times 10^{-3} \text{ GeV}^6$.

In Figs. 10(a),(b) we present the experimental and theoretical results for $\psi \equiv \arg(M^2) = 0$: in Fig. 10(a) for the choice of the obtained central value of $\langle O_6^{(V+A)} \rangle [= 0, \text{ Eq. (45b)}]$, varying $\langle aG_{\mu\nu}^\alpha G_{\mu\nu}^\alpha \rangle$ in the obtained interval (45a); in Fig. 10(b) for the choice of the obtained central value of $\langle aG_{\mu\nu}^\alpha G_{\mu\nu}^\alpha \rangle [= 0.010 \text{ GeV}^4, \text{ Eq. (45a)}]$, varying $\langle O_6^{(V+A)} \rangle$ in the obtained interval (45b). Comparison with the experimental band for $\psi = 0$ indicates a good agreement, especially for the theoretical central (full line) curve. Furthermore, comparison with the $\overline{\text{MS}}$ $\psi = 0$ curve (with its own central values 0.0059 GeV^4 and $-1.8 \times 10^{-3} \text{ GeV}^6$) indicates that the obtained central curve of the EE scheme is better.

TABLE III: Various contributions to the real part $\text{Re}B(M^2)$ of the Borel transform in the LB+bLB approach, in the (analytic) EE scheme pQCD, for various complex scales: $M^2 = 0.8 \exp(i\psi) \text{ GeV}^2$ and $M^2 = 1.2 \exp(i\psi) \text{ GeV}^2$, with $\psi = \pi/6, \pi/4, 0$. The various contributions are those indicated in Eq. (40b) and Eq. (C1), in conjunction with Eqs. (C2b) and (C10). The “LO” term is the leading term $\text{Re}(1 - \exp(-m_\tau^2/M^2))$ in Eq. (C1). For the $D = 4$ term we used the central extracted value $\langle O_4^{(V+A)} \rangle \equiv (1/6)\langle aGG \rangle = (1/6) \times 0.010 \text{ GeV}^2$; the $D = 6$ term is taken to be zero, in accordance with the central extracted value $\langle O_6^{(V+A)} \rangle = 0$.

$ M^2 $	$\arg(M^2)$	ReB: LO	ReB: LB	NLB	N ² LB	N ³ LB	sum ReB($D = 0$)	ReB($D = 4$)	ReB
0.8	$\pi/6$	1.01284	0.134652	0.000843	0.080223	0.009826	1.23839	0.025702	1.26409
1.2	$\pi/6$	0.974163	0.120230	0.000719	0.067647	0.008934	1.17169	0.011423	1.18311
0.8	$\pi/4$	1.05762	0.137521	0.000843	0.081522	0.011219	1.28873	0	1.28873
1.2	$\pi/4$	1.04447	0.124725	0.000727	0.068476	0.009661	1.24806	0	1.24806
0.8	0	0.980691	0.132679	0.0008434	0.079016	0.008587	1.20182	0.051404	1.25322
1.2	0	0.928025	0.117452	0.000715	0.067111	0.008301	1.1216	0.022846	1.14445

In Table III, we display various terms in the evaluation of the real part of the Borel transform $B(M^2)$, in the EE scheme (analytic) pQCD, with the LB+bLB approach, for various values of the complex scale M^2 : $M^2 = 0.8 \exp(i\psi) \text{ GeV}^2$ and $M^2 = 1.2 \exp(i\psi) \text{ GeV}^2$, with $\psi = \pi/6, \pi/4, 0$. These terms are based on the OPE expansion (40b), where for $D = 4$ and $D = 6$ condensates we take our central extracted values. The $D = 0$ contribution is given in Appendix C, Eq. (C1), in conjunction with Eqs. (C2b) and (C10). We denote as “LO” term the real part of the first term in Eq. (C1), i.e., $\text{Re}(1 - \exp(-m_\tau^2/M^2))$, which comes from the first (unity) term in the expansion of the full Adler function, Eq. (36b). From the Table we can see that the LB+bLB series has a rather good convergence behavior. Namely, the LB term is always significantly larger than the bLB contribution.¹³

While the theoretical curves and experimental bands shown in Figs. 9 can be interpreted as representing an extraction of the $D = 4$ and $D = 6$ condensate values Eqs. (45), the resulting theoretical curves in Figs. 10 represent our theoretical predictions for the Borel transform $B(M^2)$ for a continuous set of (positive) scales M^2 as compared to the corresponding experimental values.

The extracted values (45) can be compared with the corresponding values obtained in the literature. The spread of these extracted values is usually $\delta\langle aGG \rangle \approx \pm 0.005 \text{ GeV}^4$ and $\delta\langle O_6^{(V+A)} \rangle \approx \pm 10^{-3} \text{ GeV}^6$, if the experimental uncertainties (the spread of the grey bands) are considered to be the dominant source of the uncertainties.

In Ref. [88], where the Borel transform sum rules were used, the extracted values for the gluon condensate were $\langle aGG \rangle \approx (0.006 \pm 0.005) \text{ GeV}^4$ for the $\overline{\text{MS}}$ (truncated) and the resummed Lambert scheme pQCD and $2\delta\text{anQCD}$ model; $(0.010 \pm 0.006) \text{ GeV}^4$ for the $2\delta\text{anQCD}$ model (truncated); $(0.012 \pm 0.005) \text{ GeV}^4$ for the Lambert scheme pQCD (truncated). Here, the Lambert scheme was the scheme used for the $2\delta\text{anQCD}$ analytic model of Ref. [74] ($c_2 \equiv \beta_2/\beta_0 = -4.76$; $c_3 = c_2^2/c_1 \approx 12.74$; etc.).¹⁴ The truncated results were those based on the truncated series of the canonical Adler function $d(Q^2)$ (including $\sim a^4$); the resummed versions were those based on a $[2/2]$ -Padé-related resummation of that truncated series (cf. Ref. [88] for more details). Further, the approximate values of the $D = 6$ condensate $\langle O_6^{(V+A)} \rangle$ extracted in Ref. [88] were approximately $(-2 \pm 1) \times 10^{-3} \text{ GeV}^6$ for the aforementioned approaches ($\overline{\text{MS}}$ truncated; Lambert scheme pQCD truncated and resummed; $2\delta\text{anQCD}$ truncated), with the exception of the resummed $2\delta\text{anQCD}$ where it was $(-0.5 \pm 1.1) \times 10^{-3} \text{ GeV}^6$.

On the other hand, earlier analyses with Borel transform sum rules, performed in $\overline{\text{MS}}$ scheme, gave [97] $\langle aGG \rangle = (0.006 \pm 0.012) \text{ GeV}^4$ and [114] $(0.005 \pm 0.004) \text{ GeV}^4$.¹⁵ In Ref. [129], weighted finite energy sum rules and ALEPH 2005 data were used; the obtained values were $\langle aGG \rangle = (0.008 \pm 0.005) \text{ GeV}^4$. In the original work on sum rules, Ref. [3, 4], the value $\langle aGG \rangle = 0.012 \text{ GeV}^4$ was obtained, using charmonium physics. Application of the QCD-moment and QCD-exponential moment sum rules for heavy quarkonia, Refs. [130, 131], gave there the values $\langle aGG \rangle = (0.022 \pm 0.004) \text{ GeV}^4$ and $(0.024 \pm 0.006) \text{ GeV}^4$, respectively. Furthermore, a combined fit of the $V + A$ channel τ decay data, Ref. [96], extracted the value of α_s and, as a byproduct, the condensate value $\langle aGG \rangle = (-0.015 \pm 0.003) \text{ GeV}^4$.

Most of the analyses in the literature give $\langle O_6^{(V+A)} \rangle < 0$, cf. Refs. [88, 129, 132], suggesting a qualitative failure of the vacuum saturation approximation Eq. (44), in contrast to the result (45b) obtained here in the considered analytic

¹³ The NLB contribution is very small, principally because the NLB coefficient $T_1 = 1/12$ is very small, cf. Eq. (C2b). This is to be compared with the N²LB and N³LB coefficients, $T_2 = 103.538$ and $T_3 = 235.216$.

¹⁴ The pQCD coupling $a(Q^2)$ in the Lambert scheme is not holomorphic, and neither is in $\overline{\text{MS}}$ scheme. In comparison, in $\overline{\text{MS}}$ (with $N_f = 3$) we have: $c_2 \approx 4.47$, $c_3 \approx 20.99$.

¹⁵ In Ref. [114], in addition to the $V + A$ channel of the τ decay data of ALEPH 1998 [115], the charmonium sum rules were applied.

pQCD in the EE scheme.

VI. CONCLUSIONS

We constructed a perturbative mass independent beta function $\beta(a)$ for the QCD running coupling $a(Q^2)$ ($\equiv \alpha_s(Q^2)/\pi$) at $N_f = 3$ such that the following two restrictions are fulfilled simultaneously: (a) the correct value of the semihadronic strangeless tau lepton decay ratio $r_\tau \approx 0.203$ is reproduced, r_τ being here presently the best measured inclusive low-energy QCD quantity with strongly suppressed higher-twist contributions; (b) the coupling $a(Q^2)$ has no (unphysical) Landau singularities, i.e., it is a holomorphic function in the complex Q^2 plane $Q^2 \in \mathbb{C} \setminus (-\infty, -M_{\text{thr}}^2]$, with a threshold mass $M_{\text{thr}} \sim 0.1$ GeV. This construction was not straightforward, because the two mentioned conditions tend to mutually exclude each other. In contrast to the results of Refs. [94, 95], where the growth of the coefficients d_n of the spacelike physical quantities $\mathcal{D}(Q^2)$ due to the scheme (beta function) was out of control already at $n = 4$, we construct here beta functions which do not lead to an explosive growth of the coefficients d_n , at least up to a given chosen order n . In one case (EE scheme), we even obtained a beta function which contributes for large n to the growth of the coefficients d_n less than the leading $b = \pm 1$ renormalon contributes. Stated otherwise, the effects of our perturbative beta function did not overshadow the renormalon growth of the coefficients d_n and, at the same time, they eliminated the Landau singularities of the running coupling and allowed the reproduction of the correct value of r_τ . The attractiveness of the obtained holomorphic (analytic) QCD models is that they are perturbative, i.e., beta function $\beta(a)$ is fully described by the Taylor series in powers of a , it has no nonperturbative contributions such as $\exp(-\mathcal{K}/a(Q^2)) \sim 1/(Q^2)^M$, in contrast to the presently known analytic QCD models Refs. [39–45, 48–50, 62–75, 80, 81]. In addition, with the EE scheme (analytic) pQCD, we performed an analysis with Borel sum rules for the $V + A$ channel of semihadronic strangeless decays of τ lepton, and extracted reasonable values of the corresponding condensates: $\langle aGG \rangle = (0.010 \pm 0.005) \text{ GeV}^4$ and $\langle O_6^{(V+A)} \rangle = (0 \pm 0.001) \text{ GeV}^6$. It remains to be seen how the presented holomorphic pQCD models work in the evaluation of other low-momentum inclusive observables, such as Bjorken polarized sum rule (BSR). In contrast to the well-measured low-momentum quantity r_τ ($V + A$ channel), the BSR has strong chirality-conserving higher-twist effects at low momenta, which makes the evaluation of this quantity even in analytic QCD models more difficult [133–136].

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Appendix A: Leading- β_0 resummation of r_τ and beyond

In this Appendix, we summarize those parts of Appendices of Refs. [94, 95] that are relevant in this work. The decay ratio of the semihadronic strangeless τ lepton decays (V+A)-channel is

$$R_\tau(\Delta S=0) \equiv \frac{\Gamma(\tau^- \rightarrow \nu_\tau \text{hadrons}(\gamma))}{\Gamma(\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e(\gamma))} - R_\tau(\Delta S \neq 0) \quad (\text{A1a})$$

$$= 3.479 \pm 0.011, \quad (\text{A1b})$$

where the measured value given above is extracted from measurements by the ALEPH Collaboration [116, 117] and updated in Ref. [96]. The QCD canonic massless quantity $r_\tau(\Delta S = 0, m_q = 0)$ is obtained from this quantity by removing the non-QCD (CKM and EW) factors and contributions, and the chirality-violating quark mass contributions

$$r_\tau(\Delta S = 0, m_q = 0) = \frac{R_\tau(\Delta S = 0)}{3|V_{ud}|^2(1 + \delta_{\text{EW}})} - (1 + \delta'_{\text{EW}}) - \delta r_\tau(\Delta S = 0, m_{u,d} \neq 0). \quad (\text{A2})$$

This quantity r_τ is QCD-canonic in the sense that its (leading-twist) pQCD expansion is $r_\tau(\Delta S = 0, m_q = 0)_{\text{pt}} = a + \mathcal{O}(a^2)$. The higher-twist massless contributions in this $r_\tau(V + A)$ are very suppressed [96]. Further, it is a timelike

quantity, and can be expressed in terms of the massless current-current correlation function (V-V or A-A, both equal since massless) [137]

$$r_\tau = \frac{2}{\pi} \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + 2\frac{s}{m_\tau^2}\right) \text{Im}\Pi(Q^2 = -s) . \quad (\text{A3})$$

Using the Cauchy theorem in the complex Q^2 plane and integrating by parts results in the following contour integral form [138–144]:

$$r_\tau = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi (1 + e^{i\phi})^3 (1 - e^{i\phi}) d(Q^2 = m_\tau^2 e^{i\phi}) , \quad (\text{A4})$$

where $d(Q^2) = -d\Pi(Q^2)/d\ln Q^2 = a(Q^2) + \mathcal{O}(a^2)$ is the canonical massless Adler function, which is a spacelike QCD quantity whose expansion in powers a^n and in logarithmic derivatives \tilde{a}_n is

$$d(Q^2)_{\text{pt}} = a(Q^2) + \sum_{n=1}^{\infty} d_n a(Q^2)^{n+1} , \quad (\text{A5a})$$

$$d(Q^2)_{\text{mpt}} = a(Q^2) + \sum_{n=1}^{\infty} \tilde{d}_n \tilde{a}_{n+1}(Q^2) . \quad (\text{A5b})$$

Here, the logarithmic derivatives are defined as

$$\tilde{a}_{n+1}(\mu^2) \equiv \frac{(-1)^n}{\beta_0^n n!} \left(\frac{\partial}{\partial \ln \mu^2} \right)^n a(\mu^2) , \quad (n = 1, 2, \dots) \quad (\text{A6})$$

and are related with the powers by (repeated) application of RGE

$$\tilde{a}_2 = a^2 + c_1 a^3 + c_2 a^4 + \dots , \quad (\text{A7a})$$

$$\tilde{a}_3 = a^3 + \frac{5}{2} c_1 a^4 + \dots , \quad \tilde{a}_4 = a^4 + \dots , \quad \text{etc.} \quad (\text{A7b})$$

These relations can be recursively inverted

$$a^2 = \tilde{a}_2 - c_1 \tilde{a}_3 + \left(\frac{5}{2} c_1^2 - c_2 \right) \tilde{a}_4 + \dots , \quad (\text{A8a})$$

$$a^3 = \tilde{a}_3 - \frac{5}{2} c_1 \tilde{a}_4 + \dots , \quad a^4 = \tilde{a}_4 + \dots , \quad \text{etc.} \quad (\text{A8b})$$

Inserting the relations (A8) into the power series (A5a), we immediately obtain the coefficients \tilde{d}_n of the “modified” perturbation series (A5b) in logarithmic derivatives

$$\tilde{d}_1 = d_1 , \quad \tilde{d}_2 = d_2 - c_1 d_1 , \quad (\text{A9a})$$

$$\tilde{d}_3 = d_3 - \frac{5}{2} c_1 d_2 + \left(\frac{5}{2} c_1^2 - c_2 \right) d_1 , \quad \text{etc.} \quad (\text{A9b})$$

If the power series (A5a) is used in the contour integral (A4) and integrated for each term separately, then the obtained result is called the contour improved perturbation theory (CIPT) [142, 143]. However, since we consider here such schemes in which $a(Q^2)$ is holomorphic, there is another, probably better, approach available for the evaluation of the contour integral (A4), which involves the so called leading- β_0 (LB) resummation and the subsequent addition of three other known terms. Namely, the coefficients \tilde{d}_n (and d_n) can be written as a power series of N_f , and thus as a power series of β_0 (because $N_f = -6\beta_0 + 33/2$)

$$\tilde{d}_n = c_{n,n} \beta_0^n + c_{n,n-1} \beta_0^{n-1} + \dots + c_{n,0} , \quad (\text{A10})$$

where $\tilde{d}_n(LB) = c_{n,n} \beta_0^n$ is the LB part of the coefficient \tilde{d}_n , it is scheme independent, and is known for every n , Refs. [145, 146]. It turns out that in the series (A5b), the LB parts can be resummed [147–149]

$$d^{(\text{LB})}(Q^2) = a(\kappa Q^2) + \sum_{n=1}^{\infty} c_{n,n}(\kappa) \beta_0^n \tilde{a}_{n+1}(\kappa Q^2) \quad (\text{A11a})$$

$$= \int_0^\infty \frac{dt}{t} F_d(t) a(tQ^2 e^{\bar{c}}) , \quad (\text{A11b})$$

where $\bar{C} = -5/3$ in $\overline{\text{MS}}$ scaling convention, and the characteristic function $F_d(t)$ for Adler function is known explicitly [147]¹⁶

$$F_d(t)_{(t<1)} = 2C_F t \left[-t \ln(t) + (1+t) \ln(1+t) \ln(t) + \frac{7}{4}t + (1+t) \text{Li}_2(-t) \right], \quad (\text{A12a})$$

$$F_d(t)_{(t>1)} = 2C_F \left[\left(\frac{1}{2} + t \right) \ln(t) - t(1+t) \ln(t) \ln(1+1/t) + \left(\frac{3}{4} + t \right) + t(1+t) \text{Li}_2(-1/t) \right]. \quad (\text{A12b})$$

In the expressions above, $C_F = (N_c^2 - 1)/(2N_c) = 4/3$. If $a(Q^2)$ [and thus $\tilde{a}_{n+1}(\kappa Q^2)$] have Landau singularities at $0 < Q^2 \lesssim 1\text{GeV}^2$, the resummation (A11b) does not cure the problem of these singularities. In fact, it makes the problem formally even worse, as the integral (A11b) is then undefined (ambiguous) for any $Q^2 > 0$, due to singularities in the integrand factor $a(tQ^2 e^{\bar{C}})$ at low t . On the other hand, if $a(Q^2)$ is holomorphic, no (Landau) singularities are encountered and the LB-integral (A11b) is convergent and unambiguous.

The entire canonical Adler function

$$d(Q^2) = d^{(\text{LB})}(Q^2) + d^{(\text{bLB})}(Q^2) \quad (\text{A13})$$

consists of the LB-part (A11), and of the beyond-the-leading- β_0 (bLB) contribution whose expansion is

$$d^{(\text{bLB})}(Q^2) = \sum_{n=1}^{\infty} T_n \tilde{a}_{n+1}(Q^2) = \sum_{n=1}^{\infty} (\tilde{d}_n - c_{n,n} \beta_0^n) \tilde{a}_{n+1}(Q^2). \quad (\text{A14})$$

Insertion of the LB-integral (A11b) of Adler function into the contour integral (A4) then gives us the LB-part of r_τ

$$r_\tau^{(\text{LB})} = \int_0^\infty \frac{dt}{t} F_r^{\mathcal{M}}(t) \mathfrak{A}_1(t e^{\bar{C}} m_\tau^2), \quad (\text{A15})$$

where the superscript \mathcal{M} indicates that these are Minkowskian (timelike) quantities; \mathfrak{A}_1 is the timelike coupling

$$\mathfrak{A}_1(s) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma} \rho_1(\sigma); \quad (\text{A16})$$

and the characteristic function $F_r^{\mathcal{M}}(t)$ was obtained in Ref. [150].¹⁷ Since $-\pi d\mathfrak{A}_1(s)/d \ln s = \rho_1(s)$, integration by parts allows us to express $r_\tau^{(\text{LB})}$ as an integral over the discontinuity function $\rho_1(s) \equiv \text{Im } a(-s - i\epsilon)$

$$r_\tau^{(\text{LB})} = \frac{1}{\pi} \int_0^\infty \frac{dt}{t} \tilde{F}_r(t) \rho_1(t e^{\bar{C}} m_\tau^2). \quad (\text{A17})$$

This form is convenient here since the numerical integration of the RGE (5) gives us the values of $\rho_1(s)$ [and not $\mathfrak{A}_1(s)$], cf. Figs. 5 and 7. The characteristic function

$$\tilde{F}_r(t) = \int_0^t \frac{dt'}{t'} F_r^{\mathcal{M}}(t') \quad (\text{A18})$$

was obtained explicitly in Appendix of Ref. [94] (Appendix D of Ref. [95]); here we only reproduce it in Figs. 11, for better visualization.

However, the first three full (i.e., LB+beyond LB) coefficients d_1 , d_2 and d_3 ($\Rightarrow \tilde{d}_1, \tilde{d}_2, \tilde{d}_3$) of the Adler function are known exactly [151–156]. This means that we can add to the LB part (A15) the beyond-the-leading- β_0 contributions (bLB) of order $\sim \tilde{a}_n$ ($n = 1, 2, 3$)

$$(r_\tau)^{(\text{LB+bLB})} = r_\tau^{(\text{LB})} + \sum_{n=1}^3 T_n I(\tilde{a}_{n+1}), \quad (\text{A19})$$

¹⁶ In Ref. [147] it was argued that the expression (A11b) generates the LB part of the power expansion (A5a) when $a(tQ^2 e^{\bar{C}})$ evolves according to the one-loop RGE; in Appendix C of Ref. [148] it was shown that $a(tQ^2 e^{\bar{C}})$ can evolve according to any (N -)loop level and the integral (A11b) generates the LB part (A11a) of the “modified” perturbation expansion (A5b).

¹⁷ The quantity W_τ of Ref. [150] is related to $F_r^{\mathcal{M}}$ here via: $F_r^{\mathcal{M}}(t) = (t/4)W_\tau(t)$.

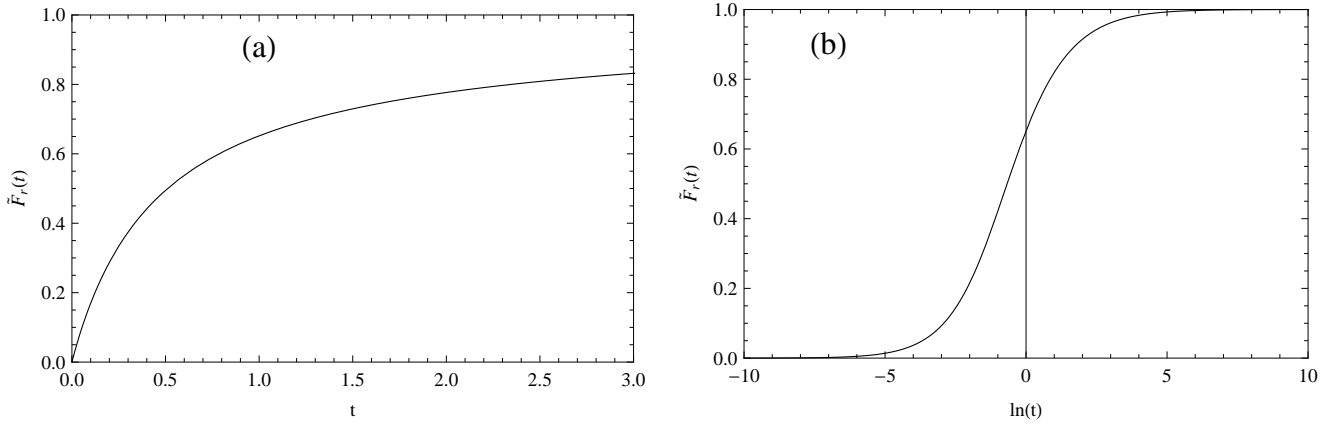


FIG. 11: The function $\tilde{F}_r(t)$ appearing in the integral (A17) of $r_\tau^{(\text{LB})}$: (a) as function of t ; (b) as function of $\ln t$.

where

$$T_n = \tilde{d}_n - \tilde{d}_n^{(\text{LB})} = \tilde{d}_n - c_{n,n}\beta_0^n, \quad (\text{A20a})$$

$$I(\tilde{a}_{n+1}) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi (1 + e^{i\phi})^3 (1 - e^{i\phi}) \tilde{a}_{n+1} (m_\tau^2 e^{i\phi}), \quad (\text{A20b})$$

and $r_\tau^{(\text{LB})}$ is given in Eq. (A17). We recall that in Eq. (A20a) the part $\tilde{d}_n^{(\text{LB})} = c_{n,n}\beta_0^n$ is scheme independent (i.e., independent of c_2, c_3, \dots), and therefore T_n has the same scheme dependence as \tilde{d}_n . We consider the expression (A19) in conjunction with Eq. (A17) as the preferred method of evaluation, and we use it for our evaluations of r_τ . Implicitly, we assume that the renormalization scale in $d(Q^2 = m_\tau^2 e^{i\phi})$ in the contour integral (A4) is $\mu^2 = Q^2 [= m_\tau^2 e^{i\phi}]$; though, other renormalization scales could be used, e.g. $\mu^2 = \kappa Q^2$ with $\kappa \neq 1$ ($\kappa \sim 1$). Furthermore, we could use for the bLB contributions in Eq. (A19) the powers a^n , i.e., the power series of $d(Q^2)^{(\text{bLB})} \equiv d(Q^2) - d(Q^2)^{(\text{LB})}$ instead of the series in logarithmic derivatives; we do not prefer this choice, because the LB part $d(Q^2)^{(\text{LB})}$ represents a (LB-)series (A11a) in logarithmic derivatives \tilde{a}_n and not in powers a^n .¹⁸

Appendix B: Coefficients of perturbation expansion in a general scheme

In this Appendix we summarize the relations between the $\overline{\text{MS}}$ scheme and general scheme coefficients in perturbation expansions of physical quantities. Perturbation expansions of spacelike observables $\mathcal{F}(Q^2)$ are usually given in the literature in the $\overline{\text{MS}}$ scheme

$$\mathcal{F}(Q^2)_{\text{pt}\overline{\text{MS}}} = \bar{a}^{\nu_0} + \bar{\mathcal{F}}_1 \bar{a}^{1+\nu_0} + \bar{\mathcal{F}}_2 \bar{a}^{2+\nu_0} + \bar{\mathcal{F}}_3 \bar{a}^{3+\nu_0} + \dots, \quad (\text{B1})$$

where \bar{a} is the coupling $a \equiv \alpha_s/\pi$ in the $\overline{\text{MS}}$ renormalization scheme ($\bar{c}_2, \bar{c}_3, \dots$) and at the canonical renormalization scale $\mu^2 = Q^2$

$$\bar{a} \equiv a(\mu^2 = Q^2; \bar{c}_2, \bar{c}_3, \dots). \quad (\text{B2})$$

The coupling in a different renormalization scheme (c_2, c_3, \dots) and at a general (spacelike) renormalization scale $\mu^2 = \kappa Q^2$ (where $\kappa > 0$, usually $\kappa \sim 1$)

$$a \equiv a(\mu^2 = \kappa Q^2; c_2, c_3, \dots) \quad (\text{B3})$$

¹⁸ If LB resummation were not used, we could use the power expansion (A5a) for the Adler function [i.e., CIPT for r_τ (A4)] since the considered holomorphic coupling $a(Q^2)$ is perturbative. On the other hand, if the considered holomorphic coupling (and beta function) were nonperturbative [$a(Q^2) \mapsto \mathcal{A}(Q^2)$], the use of the expansion (A5b) in logarithmic derivatives ($\tilde{a}_n \mapsto \tilde{\mathcal{A}}_n$ of \mathcal{A} (and its possible resummations) for the Adler function would be obligatory because otherwise the series goes out of control due to incorrect treatment of nonperturbative contributions [148, 149].

can be related to \bar{a} by use of the relations (3) and (1)-(2) (cf. Appendix A of Ref. [32], and Appendix A of Ref. [103])

$$\begin{aligned}\bar{a} = & a + a^2 \beta_0 \ln \kappa + a^3 [\beta_0^2 \ln^2 \kappa + c_1 \beta_0 \ln \kappa - (c_2 - \bar{c}_2)] \\ & + a^4 \left[\beta_0^3 \ln^3 \kappa + \frac{5}{2} c_1 \beta_0^2 \ln^2 \kappa + c_2 \beta_0 \ln \kappa - 3(c_2 - \bar{c}_2) \beta_0 \ln \kappa - \frac{1}{2} (c_3 - \bar{c}_3) \right] \\ & + \mathcal{O}(a^5),\end{aligned}\tag{B4}$$

where the notations used are those of Eq. (2), with $\beta_0 = (1/4)(11 - 2N_f/3)$ and $c_k \equiv \beta_k/\beta_0$. Substituting the expansion (B4) into the expansion (B1), and performing power expansion there in powers of a , we obtain the perturbation expansion of the physical spacelike quantity $\mathcal{F}(Q^2)$ expressed in the general scheme

$$\mathcal{F}(Q^2)_{\text{pt}} = a^{\nu_0} + \mathcal{F}_1 a^{1+\nu_0} + \mathcal{F}_2 a^{2+\nu_0} + \mathcal{F}_3 a^{3+\nu_0} + \dots,\tag{B5}$$

where the new coefficients \mathcal{F}_j are expressed by the original $\overline{\text{MS}}$ “canonical” coefficients $\overline{\mathcal{F}}_k$ in the following way:

$$\mathcal{F}_1 = \overline{\mathcal{F}}_1 + \nu_0 \beta_0 \ln \kappa,\tag{B6a}$$

$$\begin{aligned}\mathcal{F}_2 = & \overline{\mathcal{F}}_2 + \overline{\mathcal{F}}_1 (\nu_0 + 1) \beta_0 \ln \kappa \\ & + \nu_0 \left[\frac{1}{2} (\nu_0 + 1) \beta_0^2 \ln^2 \kappa + c_1 \beta_0 \ln \kappa - (c_2 - \bar{c}_2) \right],\end{aligned}\tag{B6b}$$

$$\begin{aligned}\mathcal{F}_3 = & \overline{\mathcal{F}}_3 + \overline{\mathcal{F}}_2 (\nu_0 + 2) \beta_0 \ln \kappa \\ & + \overline{\mathcal{F}}_1 (\nu_0 + 1) \left[\left(\frac{\nu_0}{2} + 1 \right) \beta_0^2 \ln^2 \kappa + c_1 \beta_0 \ln \kappa - (c_2 - \bar{c}_2) \right] \\ & + \nu_0 \left[\frac{1}{6} (2 + 3\nu_0 + \nu_0^2) \beta_0^3 \ln^3 \kappa + \frac{1}{2} c_1 (3 + 2\nu_0) \beta_0^2 \ln^2 \kappa \right. \\ & \left. + (c_2 - (\nu_0 + 2)(c_2 - \bar{c}_2)) \beta_0 \ln \kappa - \frac{1}{2} (c_3 - \bar{c}_3) \right].\end{aligned}\tag{B6c}$$

Usually we have $\nu_0 = 1$, e.g., in the case of the Adler function $d(Q^2)$ which is the underlying spacelike quantity for the (timelike) quantity r_τ , cf. Eqs. (A4)-(A5). In general, the index ν_0 may be noninteger, such as, for example, in the case of the underlying spacelike quantity for the (timelike) decay width of Higgs $\Gamma(H \rightarrow b\bar{b})$ [55, 56, 93].

Appendix C: Leading- β_0 resummation in Borel sum rules and beyond

Here we present the calculation of the $D = 0$ part of the (theoretical) Borel transform $B(M^2)$ of Eq. (41), using the LB+bLB approach described in Appendix A. Applying the contour integration (41) with the canonical Adler function $d(Q^2)$ written in the form (A13), we obtain

$$B(M^2; D = 0) = (1 - \exp(-m_\tau^2/M^2)) + B^{(\text{LB})}(M^2) + B^{(\text{bLB})}(M^2),\tag{C1}$$

where the bLB part is

$$\begin{aligned}B^{(\text{bLB})}(M^2) = & \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi d^{(\text{bLB})}(Q^2 = m_\tau^2 e^{i\phi}) \left[\exp\left(\frac{m_\tau^2 e^{i\phi}}{M^2}\right) - \exp\left(-\frac{m_\tau^2}{M^2}\right) \right]\end{aligned}\tag{C2a}$$

$$= \sum_{n=1}^3 T_n \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{a}_{n+1}(Q^2 = m_\tau^2 e^{i\phi}) \left[\exp\left(\frac{m_\tau^2 e^{i\phi}}{M^2}\right) - \exp\left(-\frac{m_\tau^2}{M^2}\right) \right],\tag{C2b}$$

where T_n are the bLB coefficients appearing in Eqs. (A14) and (A20a). The summation over n in Eq. (C2b) was truncated at $n = 3$, because only the first three coefficients d_1, d_2, d_3 ($\Rightarrow \tilde{d}_1, \tilde{d}_2, \tilde{d}_3$) are known exactly [151–156].

The LB part in Eq. (C1) is obtained in the following way. According to Eq. (40a) we have

$$B^{(\text{LB})}(M^2) = \int_0^{m_\tau^2} \frac{d\sigma}{M^2} e^{-\sigma/M^2} \omega^{(\text{LB})}(\sigma).\tag{C3}$$

According to Eq. (32), we have

$$\begin{aligned}\omega^{(\text{LB})}(\sigma) &= 2\pi \text{Im} \Pi^{(\text{LB})}(Q^2 = -\sigma - i\epsilon) \\ &= i\pi \left[\Pi^{(\text{LB})}(Q^2 = -\sigma + i\epsilon) - \Pi^{(\text{LB})}(Q^2 = -\sigma - i\epsilon) \right]\end{aligned}\quad (\text{C4a})$$

$$= i\pi \int_{Q'^2 = -\sigma - i\epsilon}^{-\sigma + i\epsilon} d(\ln Q'^2) \frac{\partial \Pi^{(\text{LB})}(Q'^2)}{\partial \ln Q'^2} = \frac{1}{2\pi i} \oint_{-\sigma - i\epsilon}^{-\sigma + i\epsilon} \frac{dQ'^2}{dQ'^2} d^{(\text{LB})}(Q'^2). \quad (\text{C4b})$$

The contour integration in the two integrals in the complex Q'^2 plane is counterclockwise along a circle of radius σ . We use in the last integral for the integrand $d^{(\text{LB})}(Q'^2)$ the integral expression (A11b), and interchange the order of integration over Q'^2 and t . This gives

$$\omega^{(\text{LB})}(\sigma) = \int_0^\infty \frac{dt}{t} F_d(t) \mathfrak{A}_1(t\sigma e^{\bar{c}}), \quad (\text{C5})$$

where $F_d(t)$ is the characteristic function of the (LB) Adler function, given in Eqs. (A12), and

$$\mathfrak{A}_1(s) \equiv \frac{1}{2\pi i} \oint_{-s-i\epsilon}^{-s+i\epsilon} \frac{dQ'^2}{Q'^2} a(s), \quad (\text{C6})$$

where $s > 0$ and the contour integration is counterclockwise in the complex Q'^2 plane. It turns out that this expression is exactly equal to the expression (A16) for the timelike coupling \mathfrak{A}_1 already encountered in Appendix A (see, for example, Refs. [51, 52, 93]). Inserting the expression (C5) into the Borel integral (C3) then gives, upon the substitution $\tau = t\sigma/m_\tau^2$ and interchanging the order of integration

$$B^{(\text{LB})}(M^2) = \int_0^\infty \frac{d\tau}{\tau} F_B(\tau; M^2) \mathfrak{A}_1(\tau m_\tau^2 e^{\bar{c}}), \quad (\text{C7})$$

where

$$F_B(\tau; M^2) = \frac{m_\tau^2}{M^2} \int_0^1 dx \exp\left(-\frac{m_\tau^2}{M^2}x\right) F_d(\tau/x). \quad (\text{C8})$$

The timelike coupling $\mathfrak{A}_1(s)$ is, according to Eq. (A16), an integral over σ of the discontinuity function $\rho_1(\sigma) = \text{Im} a(Q^2 = -\sigma - i\epsilon)$ ($\sigma > 0$). This discontinuity function is a result of the numerical integration of the RGEs (5) in the Q^2 complex plane, cf. Figs. 7. In the evaluation of $B^{(\text{LB})}(M^2)$ we would like to avoid an additional integration over σ involving $\rho_1(\sigma)$, Eq. (A16). Therefore, the trick is to apply in (C7) integration by parts in the integral over τ , and use the identity

$$\frac{d}{d \ln \sigma} \mathfrak{A}_1(\sigma) = -\frac{1}{\pi} \rho_1(\sigma) \quad (\text{C9})$$

which is a direct consequence of the identity (A16). Then the LB part of the (theoretical) Borel transform, $B^{(\text{LB})}(M^2)$, can be expressed in the following more convenient form involving ρ_1 [instead of \mathfrak{A}_1]:

$$B^{(\text{LB})}(M^2) = \frac{1}{\pi} \int_0^\infty \frac{dt}{t} \tilde{F}_B(t; M^2) \rho_1(t m_\tau^2 e^{\bar{c}}), \quad (\text{C10})$$

where the function $\tilde{F}_B(t; M^2)$, which can be called the characteristic function of the (LB) Borel transform $B^{(\text{LB})}(M^2)$, is

$$\tilde{F}_B(t; M^2) = \int_0^t \frac{dt'}{t'} F_B(t'; M^2) \quad (\text{C11a})$$

$$= \frac{m_\tau^2}{M^2} \int_0^1 dx \exp\left(-\frac{m_\tau^2}{M^2}x\right) \int_0^{t/x} \frac{d\tau}{\tau} F_d(\tau). \quad (\text{C11b})$$

Using the expression (A12) for $F_d(t)$, the integration over τ in Eq. (C11b) can be performed explicitly, and we obtain

$$\begin{aligned} \tilde{F}_B(t; M^2)_{(t < 1)} = & \frac{1}{6} C_F \left[(1 - e^{-\mathcal{K}})(21 - 2\pi^2) + \mathcal{K} \int_0^t dx e^{-\mathcal{K}x} f_1(t/x) \right. \\ & \left. + \mathcal{K} \int_t^1 dx e^{-\mathcal{K}x} f_2(t/x) \right], \end{aligned} \quad (C12a)$$

$$\tilde{F}_B(t; M^2)_{(t > 1)} = \frac{1}{6} C_F \left[(1 - e^{-\mathcal{K}})(21 - 2\pi^2) + \mathcal{K} \int_0^1 dx e^{-\mathcal{K}x} f_1(t/x) \right], \quad (C12b)$$

where $\mathcal{K} \equiv m_\tau^2/M^2$, and f_1 and f_2 are the following functions:

$$\begin{aligned} f_1(u) = & [-6 + 2\pi^2 + 6u + 3(2 + 2u - 2(1 + u)^2 \ln(1 + u)) \ln u \\ & + 6(1 + u)^2 \text{Li}_2(-1/u)] , \end{aligned} \quad (C13a)$$

$$\begin{aligned} f_2(u) = & [-21 + 2\pi^2 + 6u + 15u^2 + 3(-u(2 + 3u) + 2(1 + u)^2 \ln(1 + u)) \ln u \\ & + 6(1 + u)^2 \text{Li}_2(-u)] . \end{aligned} \quad (C13b)$$

In practice, we expanded the integrand $f_1(t/x)$ in powers of (x/t) [up to $(x/t)^{10}$] and the integrand $f_2(t/x)$ in powers of (t/x) [up to $(t/x)^{10}$], and performed the integrations over x explicitly term by term [104]. This gave us the values of the characteristic function $\tilde{F}_B(t; M^2)$ with high precision.

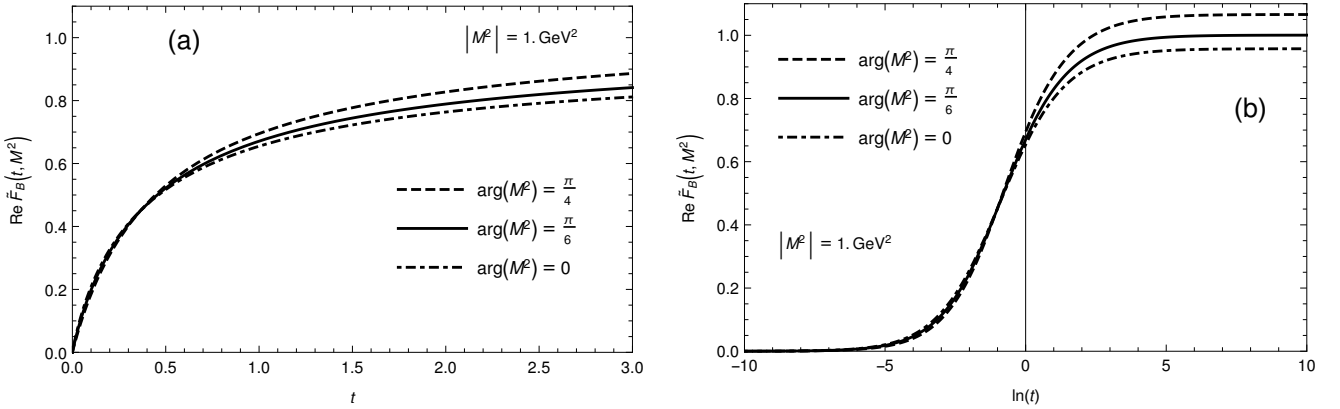


FIG. 12: The real part of the characteristic function $\tilde{F}_B(t; M^2)$ of Eq. (C12): (a) as function of t ; (b) as function of $\ln t$. Three different scales M^2 are chosen: $|M^2| = \exp(i\pi/4), \exp(i\pi/6), 1$ (in GeV^2).

In Figs. 12(a),(b) we present the real part of the characteristic function, $\text{Re} \tilde{F}_B(t; M^2)$, as a function of t and $\ln(t)$, for $|M^2| = 1 \text{ GeV}^2$ and three choices of the arguments $\psi \equiv \arg(M^2) = \pi/4, \pi/6, 0$.

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